THE HISTORY OF MATHEMATICS IN EUROPE

SULLIVAN

THE WORLD'S MANUALS

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London. England
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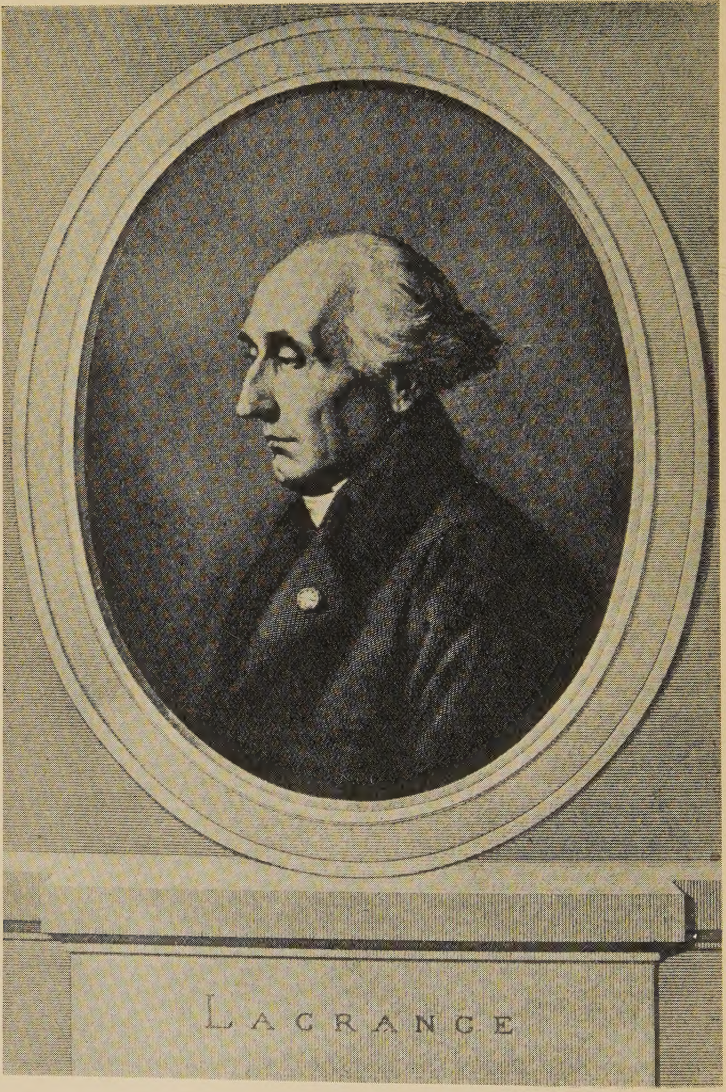
LAGRANGE

CHAPTERS IN THE HISTORY OF SCIENCE. IV

GENERAL EDITOR CHARLES SINGER

The History of Mathematics in Europe

From the Fall of Greek Science to the Rise of the conception of Mathematical Rigour

By

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PREFACE

The object of this little history is to trace, in outline, the development of mathematics from its beginnings in Europe to the invention of the differential and integral calculus. The pace of this development was determined, of course, by the number and quality of the mathematicians working at any period, and by the inherent difficulty of the subject. Another factor of great importance was the human tendency to try complicated methods before simple ones. Early mathematics was far more complicated than it need have been, partly because arbitrary divisions between similar things were established, apparently merely for the purpose of multiplying cases, and partly because the invention of a good notation seems to have presented a very difficult and lengthy task. I have thought it well to give specimens of actual solutions and sometimes of notation from early mathematicians, as this enables the reader to understand much more clearly the nature of their difficulties and the quality of their achievements.

Instead of cumbering the text with foot-note references, I take this opportunity to say that this work is based largely on Moritz Cantor's monumental treatise, *Geschichte der Mathematik*. I have also made considerable use of W. W. R. Ball's excellent *Short History of Mathematics* and am, moreover, indebted to the rapid general survey of the period here dealt with that is given in the *History of Mathematics* by Florian Cajori.

I have omitted direct discussion of Greek mathematics as that subject has already been treated in this series in the work of Professor Heiberg.

The illustrations have been selected by Dr. Charles Singer.

J. W. N. SULLIVAN.

FLORENCE, 1924.

CONTENTS

LIST OF ILLUSTRATIONS

INTRODUCTION

It is convenient to keep the old classification of mathematics as one of the sciences, but it is more just to call it an art or a game. The significance of any human pursuit becomes clearer the longer mankind endures, and in calling mathematics an art or a game we are passing no judgement on its *value*. We are merely saying that, unlike the sciences, but like the art of music or the game of chess, mathematics is a free creation of the mind, unconditioned by anything but the nature of the mind itself. And we may agree that the honourable estimation in which the mathematician has always been held by modern nations is probably justified. His activity probably has a meaning because it is profoundly related to human destiny. But the old conception of the significance of mathematics is certainly wrong. We no longer believe, with Pythagoras, that in the properties of numbers are to be found the secrets of the Universe, and we no longer believe that the attributes of God can be deduced from the *necessary* nature of the axioms of Euclidean geometry. While such beliefs were held mathematics was certainly a science, and might even rank with the Queen of the sciences. But we now know that there is nothing necessary in any of the fundamental postulates or definitions of any branch of mathematics. They are arbitrary: which does not mean that their choice was *psychologically* arbitrary, but that it was *logically* arbitrary.

The arbitrary nature of mathematical postulates was not generally understood by mathematicians until well on into the nineteenth century. It was the creation of the non-Euclidean geometries which necessitated a radically different estimation of the status of mathematical entities. The geometry of Euclid became one of an infinite number of geometries, all perfectly logical and self-consistent. Which geometry could be most simply applied to the external world became a question to be decided by experiment. The fundamental entities to which mathematical reasoning is applied exist purely in virtue of their definition. Such entities as point, three, straight line, &c., are free creations, and all we need ask of them is that the members of any given group are defined so as to be consistent with one another. This discovery immediately cleared up a number of difficulties. Early mathematicians were much puzzled by, for instance, negative numbers. It was thought that a negative number was somehow less real than a positive number; the one existed in the nature of things, but the other did not. Various attempts to find analogues to negative numbers in the constitution of the world were made. Thus it was suggested that a negative number corresponded to the state of 'owing'—a conception which, even thirty years ago, had not completely vanished from school books. Imaginary numbers also gave rise to much speculation. Every new mathematical entity had to be made, somehow, a part of the universe in a sense quite other than as a mere product of the mathematician's mind. Thus even Leibniz found metaphysical truths expressed by the properties of infinite series.

With the realization of the true logical status of mathematics comes a clearer view of its humanistic value. It is an independent art, but, as it happens, it can be applied to the interpretation of nature. Considered as an art, it appeals to very few, and, on this ground alone, it is difficult to see why the ordinary run of men should pay more homage to a mathematician than to a chess champion. Yet it is not clear that mathematics is revered chiefly for its usefulness to the physical sciences. In the face of this mysterious benevolence on the part of mankind at large the mathematician can only express his gratitude. Certainly the history of mathematics does not explain this general esteem.

It was only rarely that an important personage, such as the *stupor mundi* Frederick II, had a taste for it. Even its connexion with the fears and hopes of mankind, via astrology, is not sufficient to explain the position accorded to it. It seems probable that man is aware, more or less obscurely, that there are powers and qualities of the mind which have a value apart from that which is directly utilitarian. Such elements probably form an essential part of the developing human consciousness, and for that reason have an increasing significance.

With the invention of mathematics the mind entered upon the exploration of a new kingdom, full of the promise of new beauty and new power. The language of aesthetics is never far to seek in the writings of a great mathematician, and every new generalization gives an increased sense of power. This region, created by the mind, seems ever more varied and ever less limited. One discovery suggests another ; it does, in fact, *create* another. The impulse which created mathematics, although later in its manifestation than the impulses which created the other arts, is not less living or persistent. In the history of mathematics we watch this persistent desire surviving, although barely surviving, neglect and discouragement. Remote as it often appears to be from all else that interests men, mathematics yet satisfies some deep human need. For this reason its history is worth studying. It appears as a strangely isolated activity. Manners and customs, religions and philosophies, are not reflected in its development. Even music, the most independent amongst what are generally reckoned as the arts, has been much more influenced by its *milieu* than has mathematics. Mathematics has a self-creating energy ; the direction of advance is determined by the point that has been reached, as when elementary considerations of form led men from equations of the second degree to equations of the third degree, and then to equations of the fourth degree and endeavours to carry the process on to equations of the fifth and higher degrees.

Similarly, from conic sections to curves of higher orders the progress was equally straightforward. With the development of the physical sciences, it is true, branches of mathematics were called into existence by physical problems, but we may say that in practically all cases the actual path of development followed by mathematics has been determined by the mathematician's sense of form—a consideration or criterion which is purely mathematical. Mathematics is the most completely autonomous of all human activities. It is thus, from this point of view, the purest of the arts, although it differs from the other arts in that the work of each period is taken up as an integral part of the work of the next, and thus comes to exist, not independently, but absorbed into a greater wholeness and beauty. Although, therefore, a history of mathematics is largely a history of discoveries which no longer exist as separate items, but are merged into some more modern generalization, these discoveries have not been forgotten or made valueless. They are not dead, but transmuted.

I

The Dark Ages in Europe

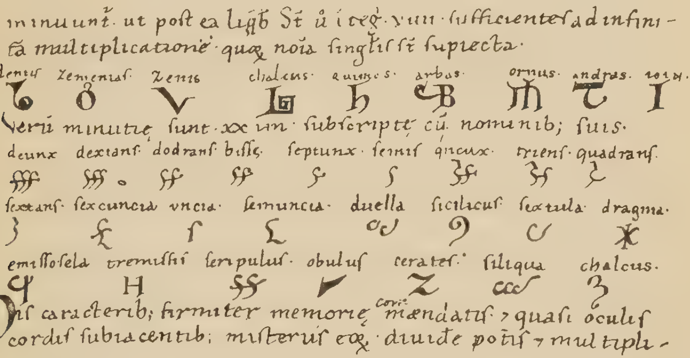
The Passage of Greek Mathematics to the Arabs

By the time the sixth century A.D. was reached the great traditions of Greek learning had almost entirely died out. The Roman spirit, narrowly practical and with very little intellectual curiosity or imagination, was inimical or indifferent to all those free activities of the human mind which had resulted in the wonderful wealth of Greek science and mathematics. The shadow of the mighty Roman Empire spread like a cold blight over the scientific life of the ancient world; in that heavy, unresponsive atmosphere the great adventure begun by the Greeks could not continue; men relapsed once more into the old slavery to the purely practical and sensual life. In this respect the Romans resembled the other peoples of antiquity; it was the Greeks who were the exceptions. There is no evidence that any one had any conception of that distinctive mental activity which we call mathematical reasoning before the time of Pythagoras. The mathematical formulae which were in use amongst the Egyptians, for example, were chiefly connected with land-surveying problems, and were evidently obtained empirically. They are usually inaccurate and are nowhere accompanied by proofs. The whole notion of a mathematical proof, i. e. of the logical deduction of results from postulates, seems to have been foreign to ancient minds, with the single exception of the Greeks. The discovery of this unsuspected possibility of the mind was one of the greatest steps forward in the development of the human consciousness. But the Romans had no share of this strange intuition and desire, and under their rule this new kingdom of the mind lay completely neglected. It was not until the sixth century, when the dominance of Rome was at an end, that a few pitiful scraps of the Greek mathematical learning were made accessible to the European mind.

Boethius (475–526), a member of one of the most illustrious families of Rome, was exceptionally interested in Greek literature and science. He published a *Geometry*, consisting of the enunciations of the first book of Euclid, and of some propositions from the third and fourth books. He gave proofs of the first three propositions, and numerous practical applications of the theorems. He also published an *Arithmetic*, essentially a translation of a text-book by the Syrian *Nicomachus*, a work which enjoyed a great reputation for many years. These books, together with the writings of another Roman, *Cassiodorus* (490–585), were the chief sources of mathematical learning accessible to students of the early middle ages.

Even this degree of learning was not common ; it was only the more erudite that mastered Boethius. We can summarize this period by saying that mathematical learning remained in this rudimentary state until the beginning of the twelfth century. The chief cause of this state of affairs, apart from perpetual wars, was that the Church had turned men's attention to other subjects. Mathematics, in comparison with theology, was of no importance, and although mathematics formed part of the recognized courses of instruction, the intellectual trend of the age was such as to prevent it from becoming a popular subject. Even so, a mathematical genius might have found sufficient stimulus in that early teaching to realize himself, but apparently no mathematical genius happened to appear during those years. There is not one of the exceptional intellects of that time that shows any great mathematical ability. Perhaps one of the greatest mathematical feats of that time was performed by the famous *Gerbert* (died 1003), who became Pope in 999. In the course of his work on *Geometry* he solved the problem, very difficult at that time, of determining the sides of a right-angled triangle whose hypotenuse and area are given.

But although mathematics was at this low ebb in Europe, the great triumphs of the Greek intellect were not lost to mankind. The Arabs, under the influence of a great religious impulse, had very rapidly become a conquering race. But the change from their nomadic life to the life of cities seems to have made them subject to diseases hitherto unknown to them, and able physicians became in great demand. The best physicians of that time were Greeks and Jews trained in the Greek traditions, and these



Portion of a twelfth-century MS. at the British Museum (Arundel 343) showing the Boethian numerals and signs for fractions, weights, and measures

men had an extensive knowledge of Greek achievements in other sciences than medicine. There were also various small Greek schools in the conquered territory. The Arab caliphs seem to have had singularly open and inquiring minds. Able specialists were invited to the court at Bagdad, irrespective of nationality or religious belief, and all that they had to teach was eagerly absorbed. Directly the Arabs found that the Greeks were in possession of scientific and mathematical learning much superior to their own they set themselves to acquire it. In this way, by the year 800, the works of Hippocrates, Aristotle, and Galen were translated into Arabic by the order of the famous caliph *Haroun Al Raschid.* His successor, *Al Mamun* (813–33), sent a commission to Constantinople to acquire as many scientific works as possible, and employed a staff of translators to translate them into Syriac and Arabic. In the space of one hundred years, by the end of the ninth century, the Arabs were in possession of most of the masterpieces of Greek science. They had translations of Euclid, Archimedes, Apollonius, Ptolemy, and others. But this does not exhaust the mathematical knowledge accessible in Arabic. Bagdad, which reached out to Greece in the west, reached out to India in the east, and absorbed, besides the mathematical learning of the Greeks, the mathematical learning of the Hindoos.

The Hindoo Mathematicians

The Hindoos are the only people of antiquity, besides the Greeks, who display any marked mathematical ability, and it is interesting to note that these two peoples illustrate the two classes, the geometrical and the analytical, into which, according to Henri Poincaré, all mathematicians may be divided. The Greeks were geometricians; the Hindoos, analysts. Hindoo geometry is comparatively poor, but in arithmetic and algebra they were distinctly superior to the Greeks. Greek numerical symbolism was sufficiently clumsy to make the ordinary operations of arithmetic of considerable difficulty. Our present symbolism we owe to the Hindoos, and their principle of position, and the introduction of the zero, must be regarded as first-rate inventions. The history of Hindoo symbolism is obscure; it is probable, however, that number characters were in use by 150 B.C. and the zero was probably introduced before A.D. 600.

The writings of the first Hindoo mathematician of note, *Aryabhata* (born 476), seem to imply that he was in possession of the principle of position and of the zero. He had the habit, common to Hindoo mathematicians, of expressing his results in verse, and the language is often very obscure. His chief work is the *Aryabhathiya*, of which the first three parts are devoted to astronomy and spherical trigonometry, and the fourth part to arithmetic, algebra, and plane trigonometry. The mathematical ability shown is very considerable. In algebra he sums the first n natural numbers and the second and third powers of these numbers. He gives the general solution of a quadratic equation. In dealing with indeterminate equations he exhibits the peculiar subtlety of the Hindoo mind. Diophantus was content with a single solution of such equations, and his amazing ingenuity was shown in dealing with each equation as a particular case. The Hindoos endeavoured to find all possible integral solutions. Thus Aryabhata gives the integral solutions to linear equations of the form $ax \pm by = c$ where a, b, c are integers. His solution is essentially the same as the one given by Euler. In trigonometry he gives a table of natural sines for angles from 0° to 90°, proceeding by multiples of $3\frac{3}{4}°$. In this result he assumes 3·1416 for the value of π. His geometry was much inferior to his analysis, however, and a number of his geometrical propositions are wrong.

The next great Hindoo mathematician of whom we have any record is *Brahmagupta*, who flourished during the first half of the seventh century. He was familiar with the work of Aryabhata, whom he often quotes, but there seems little doubt that much of his work is original, although it must be supposed that some mathematical advance had been made during the hundred years separating these two writers. Brahmagupta's work, written also in verse, has two chapters on arithmetic, algebra, and geometry. Of particular interest is his solution of the indeterminate equation of the second degree, $nx^2 + 1 = y^2$. He gives as its solution

$$x = 2t/(t^2 - n) \text{ and } y = (t^2 + n)/(t^2 - n).$$

This same equation was set by Fermat as a challenge problem a thousand years later, when precisely the same solution was found by Lord Brouncker. In his geometry Brahmagupta is weaker, although he still shows considerable ability. Thus he proved the Pythagorean theorem, and also determined the surface and volume of a pyramid and a cone.

The writings of other Hindoo mathematicians are extant, of whom the most famous is *Bhaskara* (born 1114), but it is doubtful whether they had any influence on the subsequent history of European mathematics. It is certain that some of the most valuable results obtained by the Hindoos were never imported into Europe, and later European mathematicians had to discover them for themselves. What is of interest to our present purpose is the fact that by the end of the ninth century the Arabs were in possession of the elements of modern arithmetic and algebra, created by the Hindoos, and of the very considerable body of geometrical knowledge which had been created by the Greeks.

The Beginnings of European Mathematics

While it would not be just to say that the Arabs merely conserved the mathematical learning they had obtained from the Greeks and Hindoos, it is nevertheless true that Arabian mathematicians were, on the whole, more learned than original, and that they made no mathematical discovery of the first importance. For some generations they were fully occupied in assimilating the great wealth of material they had acquired, and when, after this period of preparation, they began to do original work, it was chiefly of the kind that may be described as corollaries to the great theorems of their foreign teachers. It is worth noting that the work of the early Arab mathematicians makes no clear division between arithmetic and algebra. Like the Hindoos, and unlike the Greeks, they do not seem to have been bothered by subtle considerations respecting the relations of discrete and continuous quantities. The Greeks gave geometrical proofs of algebraic propositions, for a line or an area may represent a continuous quantity; considerations of this sort did not hamper the Hindoos.

The first Arabian mathematician of note, commonly called *Alkarismi*, wrote an algebra in 830 which influenced many subsequent writers. It is founded on the treatise of Brahmagupta, but for some of the proofs he has adopted Greek geometrical methods. He expounds two general methods of treating equations, *al-gebr*, from which our word algebra is derived, and *al mukabala*. The method *al-gebr* simply means transferring negative quantities to the other side of the equation, and the method *al mukabala* means the uniting into one term of similar terms. He divides quadratic equations into five classes:

$$ax^2 = bx,\ ax^2 = c,\ ax^2 + bx = c,\ ax^2 + c = bx,\ ax^2 = bx + c,$$

where a, b, c, are positive numbers. A little geometry is also contained in the treatise. He gives the theorem of Pythagoras for the simplest case, and also calculates the areas of the triangle, parallelogram, and circle. He knows the following approximations to π, viz. $3\frac{1}{7}$, $\sqrt{10}$, and 62832/20000.

Following on *Alkarismi* came *Tabit ibn Korra* (836–901), astronomer, mathematician, and linguist. His chief value for mathematics was in his translations of Euclid, Archimedes, Apollonius, and Ptolemy. A fragment of his original writings has been preserved, consisting of geometrical solutions of cubic equations. This branch of analysis received considerable attention from the Arabs, but their work did not influence European mathematics. Some of their methods were discovered much later by European mathematicians. One of the ablest of these Arabian analysts was *Alkayami*, who flourished about A. D. 1000. He obtained a root of several cubics by using conic sections, and also solved a biquadratic. His contemporary, *Alkarki*, was also an analyst of ability. He summed the squares and also the cubes of the first n natural numbers and solved equations of the form $ax^{2n} \pm bx^n \pm c = 0$. His work on indeterminate equations is purely Diophantine; he appears to have been quite unacquainted with the Hindoo achievements in this field. And a very surprising fact is that a work on arithmetic by the same author shows no knowledge of the Hindoo numerical system.

With these writers Arabian mathematics reached its highest point. Between A. D. 1100 and 1300 came the Crusades and the assaults of the Mongols. Arabian science declined, although their schools continued to exist until the fifteenth century. It was during the decline of Arabian science that the Arab works became known to Europe. This knowledge first came via the Moors in Spain who, in their colleges at Granada, Cordova, and Seville, possessed the Arabic translations from the Greek, together with the writings of the chief Arabic mathematicians. The acquisition of this knowledge was at first a task of very great difficulty, and required physical courage as much as mental ability, for the Moors jealously guarded their knowledge from Christians, and attempts to evade their decrees were attended by very considerable risks. Amongst the first men to undertake the adventure was an English monk, *Adelhard* of Bath, who disguised himself as a Mohammedan student and attended lectures at Cordova about 1120. He also travelled in Asia Minor and Egypt, carrying his life in his hand, but determined to master the science of the Mohammedans. He managed to acquire a copy of Euclid's *Elements* and translated this into Latin. He also translated the astronomical tables of Alkarismi and, probably, his arithmetic. Adelhard's translation of Euclid was not superseded in Europe until the sixteenth century, when the Greek text became available.

Other translators of the twelfth century were the rabbis *Ben Ezra* and *Johannes Hispalensis*, the latter of whom was converted to Christianity. A little later (1114–87) *Gerard* of Cremona, eager to become acquainted with Ptolemy's *Almagest*, went to Toledo and there learnt Arabic and translated it in 1175. His zeal for knowledge, and his industry, led him to translate over ninety Arabic works, including the books of Euclid and the algebra of Alkarismi.

The history of this translation activity brings us to the dawn of the thirteenth century, when the mathematical mind of Europe first showed its strength. By the labours of these devoted translators the West was now in possession of the Hindoo numerical notation, the algebra of linear and quadratic equations, the geometry of Euclid, and the astronomy of Ptolemy. This body of knowledge was a legacy from the Greek and the Hindoo ; it had been conserved by the Arabs and was now accessible to the European. After a long period of almost suspended animation mathematics was once more to be cultivated by minds having the vigour and subtlety necessary to carry on to still more splendid triumphs the adventure begun so magnificently by the Greeks.

The First European Mathematicians

Leonardo of Pisa and Jordanus Nemorarius

WITH the opening of the thirteenth century there appear two mathematicians in Europe who may fairly be called great, *Leonardo of Pisa* and *Jordanus Nemorarius*, the first of whom was a merchant and the second a monk. It is probable that Jordanus wrote a little before Leonardo, and it has been argued that his influence was greater. His work was not known by scholars until quite recently, however, and Leonardo has usually been taken to be the dominating figure of this century and, indeed, of the next. Leonardo was born at Pisa about 1175, his father being one Bonacci, a merchant. As a mere boy Leonardo acquired a strong taste for mathematics, and during his later extensive business travels in Egypt, Syria, Greece, and Sicily he mastered whatever was known of mathematics by the people with whom he came in contact. When about twenty-five years of age he returned to Italy and two years later published his famous *Liber Abaci*, the greatest mathematical treatise which had yet appeared from a European. A second edition, which is the work as known to us, appeared in 1228.

This work may be said to have introduced the Hindoo numeral notation into Europe, for, in explaining the advantages of this notation, Leonardo expressly says that he finds it more convenient than the one in use amongst his countrymen. The book is divided into fifteen sections and deals with arithmetic and algebra. A very large number of examples are worked out.

Leonardo's favourite method of solving many problems is by the method of 'false assumption', which consists in assuming a solution and then altering it by simple proportion as in the rule of three. We give as an example Leonardo's solution of a problem set him by a magister in Constantinople. The problem runs: If A receives from B 7 denare, then A's sum is fivefold B's; if B receives from A 5 denare, then B's sum is sevenfold A's. How much has each? In order to apply the rule of false assumption Leonardo recasts the problem. Draw a straight line *aegdb* and let *ag* represent A's sum and *gb* B's sum. Then the total sum is *ab*. Further, let *eg* represent 5 and *gd* 7. Then, by the conditions of the problem, $ag+gd$ is five times what is left to B, i.e. is five times *db*. Therefore *db* is $\frac{1}{6}$ of the total sum. By similar reasoning *ae* is $\frac{1}{8}$ of the total sum. Therefore $db+ae=\frac{1}{6}+\frac{1}{8}$ of the total sum, and this subtracted from the total sum leaves $eg+db=5+7=12$ denare. Leonardo now applies the method of false assumption. Suppose that the total sum is 24. Then $\frac{24}{6}+\frac{24}{8}=7$, and this subtracted from 24 leaves 17. But the correct remainder is 12. The sum is therefore $\frac{12}{17}$ of 24, and the numbers 4 and 3 assumed for *db* and *ae* must be reduced in the same proportion. Therefore *db* is really $\frac{12}{17}\times 4=2\frac{14}{17}$ and *ae* is $\frac{12}{17}\times 3=2\frac{2}{17}$. A's sum, therefore, is $2\frac{2}{17}+5$ and B's $2\frac{14}{17}+7$.

Besides this extraordinary solution Leonardo uses the *Regula recta*, i. e. he uses what is to us the obvious method of expressing the conditions as two first order equations. But, except in one place in his work, Leonardo does not employ letters as algebraic symbols. He calls the unknown quantity *res*. And the absence of signs for ordinary algebraic processes must have been a great hindrance. It must be remembered that many apparently easy problems solved by early mathematicians were really of considerable difficulty purely because a good algebraical symbolism had not been invented. The eye received no aids; the statements were often in words; and it is only by a study of these early proofs that one fully realizes how extremely important a good notation is for the development of mathematics. The section in which occurs the problem we have given starts with the important summation formulae

$$a+(a+d)+\ldots+(a+(n-1)d)=(a+(a+(n-1)d))\frac{n}{2}$$

and

$$a^2+(2a)^2+\ldots+(na)^2=\frac{na(na+a)(na+(na+a))}{6a},$$

and examples of the use of these formulae are given. In a later section Leonardo deals with the extraction of roots and succeeds in obtaining approximate values for both square and cube roots. In algebra he deals with simple and quadratic equations.

This work seems to have attracted, soon after its appearance, the notice of highly placed personages. The Emperor Frederick II was very interested in science, and it was largely through his efforts that the work of Arab mathematicians was made accessible to Europe. Through the good offices of an astrologer, Dominicus, Leonardo was invited to the Court, and his visit was made the occasion of a mathematical tournament. John of Palermo, attached to the Emperor's suite, set the problems. The first problem was to find a number such that its square, when increased or decreased by 5, would remain a square. Leonardo gave the answer 41/12, which fulfils the conditions. The next problem was to solve the cubic equation $x^3+2x^2+10x=20$. The solution of cubics was, of course, unknown at that time, but Leonardo showed great ability in obtaining a very close approximate solution. His solution, expressed in decimal notation, is 1·3688081075, and is correct to nine places. The third problem was easier. Three men possess a sum of money, their shares being $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. Each one of them takes some money from the pile until the whole sum is taken. Then the first man puts back $\frac{1}{2}$ of what he had taken, the second $\frac{1}{3}$, and the third $\frac{1}{6}$. When the total so put back is divided equally amongst the three men it is found that each man then possesses what he is entitled to. What was the total sum, and how much did each man take from the original pile? Leonardo's solution is worth quoting for its elegance. Call the third part of the total sum put back x, and the original total sum s. Each man, by taking x, gets what he was entitled to, i.e. the sums $\frac{s}{2}, \frac{s}{3}, \frac{s}{6}$. Therefore the three men possessed, before taking x,

$$\frac{s}{2} - x, \ \frac{s}{3} - x, \ \frac{s}{6} - x.$$

But they possessed these sums after putting back $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ of what they had first taken. These sums are therefore $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$ of what they had first taken. Therefore what they had first taken were

$$2\left(\frac{s}{2} - x\right), \ \frac{3}{2}\left(\frac{s}{3} - x\right), \ \frac{6}{5}\left(\frac{s}{6} - x\right).$$

And these sums together make s. Hence

$$s = s - 2x + \frac{s - 3x}{2} + \frac{s - 6x}{5} \text{ or } 7s = 47x.$$

The problem is therefore indeterminate. One solution is $s = 47$, $x = 7$, and the sums taken by the men from the original pile are then 33, 13, 1. This is Leonardo's solution given in modern notation. Leonardo called x *res* and s *tota communis pecunia*. These solutions were published by Leonardo in two remarkable works, the *Liber quadratorum* and the *Flos*. Besides these works Leonardo published a *Practica geometriae* which also shows great ability.

Excepting his contemporary, Jordanus, no mathematician of Leonardo's ability appeared for two hundred years. Even his treatise, the *Liber Abaci*, although it had great influence, seems not to have been fully mastered during the next two centuries. Besides his great natural ability Leonardo seems to have been gifted with a stubborn, persevering temperament. When he learned or invented a method he set himself also to exhaust it. Hence the great number and variety of problems in his writings.

Contemporary with Leonardo was a German mathematician of scarcely inferior ability, called *Jordanus Nemorarius* or *Jordanus Saxo*. Very little is known about his life, and no precise dates can be assigned to his works. We know, however, that he became general of the Dominican order in 1222, and that at this time the order was expanding rapidly. It is probable, therefore, that all his mathematical works were written before this date. He wrote on arithmetic, algebra, and geometry, and probably also on mechanics. In his arithmetic he shows that he knows the expression

$$(a+b+c+\dots)^2 = a^2+b^2+c^2+\dots+2ab+2ac+\dots+2bc+\dots$$

A striking feature of Jordanus's algebra is his use of letters for magnitudes. This was an important step forward in the evolution of algebraic symbolism, although it does not seem to have had much influence on subsequent writers. The same device had to be invented again, more than once, before it was generally adopted. But although Jordanus used this device it often does not add appreciably to the clarity of his statements. The absence of signs for the ordinary operations of arithmetic (except addition, which Jordanus denotes by juxtaposition) meant that every multiplication or subtraction resulted in a fresh letter, so that very great concentration is required to follow even a moderately long chain of reasoning.

It is perhaps in his *geometry* that Jordanus shows most ability. He discusses the properties of triangles, ratios of straight lines, ratios of triangular areas, arcs and chords of circles, regular polygons, and a few special problems, such as the trisection of an angle. In his algebra we find simple and quadratic equations, sometimes with more than one unknown quantity. His work, considered as a whole, is that of a man of considerable learning and ability. He has not quite the brilliance of Leonardo, and in one or two respects he is something of a pedant. Neither Leonardo nor Jordanus belong to the first rank of mathematicians, but in the long centuries of dullness in the midst of which they lived their works, besides their real merits, have the extra charm which is given to an oasis by the surrounding desert.

Other writers on mathematics during the thirteenth century require but the briefest mention. Even *Roger Bacon*, one of the most original geniuses of all time, is of very little importance in the history of mathematics. That he fully perceived the value of mathematics is unquestioned, but his work shows that his own ability in this direction was not very great. Mention must be made of *John of Holywood* or *Johannes de Sacrobosco*, who taught mathematics in Paris until his death there in the middle of the thirteenth century. His best known work, *De sphaera mundi*, was popular for three hundred years. It is merely a compilation made up chiefly of extracts from Ptolemy's *Almagest*, together with some from Arabian astronomers. He published also a work on arithmetic. This work is merely a collection of rules; it contains no proofs and no examples. It is a sufficient indication of the state of mathematics at that time that these works were regarded as important, and commentaries were written on them.

Another writer of this period was *Johannes Campanus*, who published an edition of Euclid's *Elements*. The edition includes a commentary which shows some mathematical ability, but to what extent these comments were original with Campanus it is impossible to say. Such were the mathematical writings of the thirteenth century. It is a little curious that this century, one of the most interesting in history, and characterized by the enormous effort which was made to gather together and to systematize all knowledge, should be so barren mathematically. It is true that *Vincent de Beauvais*, in his gigantic encyclopaedia, finds a place for mathematics, but the treatment is quite superficial. The acutest mind of that age, *Thomas Aquinas*, did not concern himself with mathematics. The only man of the time who fully realized its importance was Roger Bacon. Theology was the queen of the sciences, and most of the best minds of the age devoted their energies to the subtleties of scholastic philosophy.

The Fourteenth Century

Mathematics continued to languish during the fourteenth century. Indeed, no mathematical writings of real importance appeared after those of Leonardo of Pisa until the Renaissance. But we may notice two writers of the fourteenth century, since their work contained ideas which were to receive future development. The earlier of these was *Thomas Bradwardine*, Archbishop of Canterbury, born 1290 and died 1349. He wrote an *Arithmetica speculativa*, a *Geometria speculativa*, and a work entitled *De proportionibus velocitatum*. But of greater interest than these is a manuscript called *Tractatus de continuo*, which discusses the nature of continuous and discrete quantity. In connexion with this is discussed also the notions of the infinitely large and the infinitely small. Questions are here raised which were to occupy mathematicians for centuries, and which did not receive a final solution until very recent times. It is not suggested that Bradwardine initiated these questions or that his treatment of them was even known to subsequent mathematicians; his interest to us lies in the fact that he was here concerned with ideas of real mathematical importance.

The other writer who deserves mention is *Nicholas Oresmus* (1323–82), who wrote an *Algorismus proportionum* in which the notion of fractional indices is introduced. He enunciated the rules

$$(a^m)^{\frac{p}{q}} = (a^{mp})^{\frac{1}{q}}; \quad \frac{a}{b^{\frac{1}{n}}} = \left(\frac{a^n}{b}\right)^{\frac{1}{n}},$$

and several others, although not, of course, in modern notation. In this part of his work Oresmus was much in advance of his time.

During the fourteenth century arose a contest as to the seat of the Papacy, whether it should be at Avignon or at Rome. From 1305 to 1377 only Frenchmen were elected Popes, and had their seat at Avignon. The Church was divided into two parties, those for and against Rome as the seat of authority, and this split extended to the universities. German teachers and scholars in Paris were on the side of Rome, and this difference of opinion caused many of them to return to Germany, where several universities were founded at this time. It is interesting to know the standard of university mathematics during the latter part of the fourteenth century. At Prague there were courses on the sphere, the first six books of Euclid, arithmetic, and Ptolemy's *Almagest*. At Vienna was taught the theory of proportions, perspective, the first five books of Euclid, and the measurement of areas. The teacher read through works on these subjects with the student, but there were no examinations. Degrees were granted on the student taking an oath that he had listened to the required course of exposition. It is probable, therefore, that the mathematical knowledge actually possessed by students was distinctly inferior to that theoretically obtainable.

3

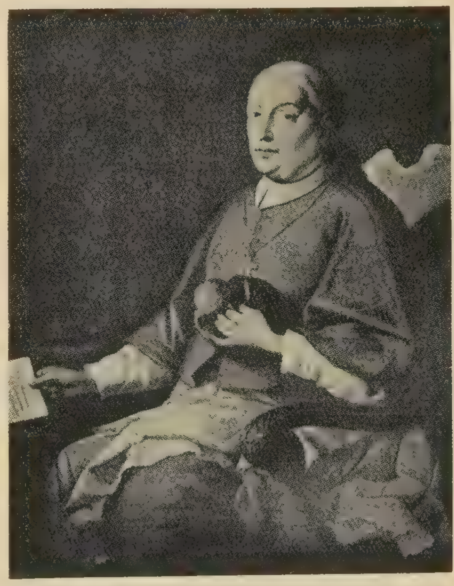
The Fifteenth Century

Regiomontanus

During the fifteenth century began that great intellectual awakening, that increased activity in all branches of science and art, which we call the Renaissance. As the Eastern Empire dwindled in size and power refugees from Constantinople came to Italy bearing with them the magnificent treasures of Greek civilization. The invention of printing enabled knowledge to be disseminated with unprecedented rapidity. Henceforth mathematics, in common with the other sciences, steadily progressed. The outstanding mathematician of this century was *Johann Regiomontanus*, whose tutor was *George Purbach* (1423–61). Besides teaching mathematics Purbach lectured on the chief Latin poets. His chief work was the preparation of a Latin translation of the *Almagest* from the original Greek text, but the task was interrupted by his death. Among his mathematical works may be mentioned an arithmetic and a table of natural sines.

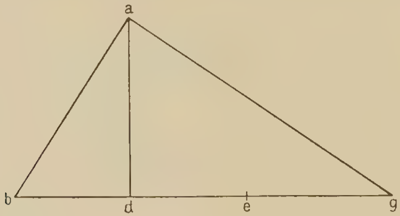
Amongst his contemporaries the Cardinal, *Nicholas Cusanus* (1401–64), so called from his birth at the small town of Cues near Trèves, deserves some mention. The library of Nicholas remains to this day almost intact at Cues, and gives a good idea of the reading of a learned man of that age. Nicholas concerned himself with mathematics only incidentally, and the chief problem that engaged his attention was the quadrature of the circle. He regarded the circle as a polygon possessing an infinite number of sides. In the course of these attempts he obtains various values for π. In his endeavour to draw a circle whose circumference shall be equal to the sum of the sides of a triangle he obtains for π the value $3{\cdot}142337$, which is more accurate than the Archimedian value $3\frac{1}{7}$. Cusanus also wrote on the notion of mathematical infinity.

We now come to consider the work of *Johannes Müller* (1436–76), or, as he was commonly called, *Regiomontanus*, from his birthplace at *Königsberg*, that is *king's mountain*, or in Latin *regius mons*. Regiomontanus was the greatest mathematician that Europe had yet produced since the death of Leonardo of Pisa. As a mere child Regiomontanus became a pupil of Purbach at Vienna, and lectured while he was yet too young to be granted a degree. Purbach, who was the friend as well as the tutor of the young man, entrusted him with the task of finishing the translation of the *Almagest*. In order to master Greek with the thoroughness the task demanded, Regiomontanus went to Rome, and copied out several Greek manuscripts, most of which, although not all, were mathematical. During the following years Regiomontanus travelled incessantly in Italy and Germany until, in 1471, he settled at Nuremberg, where he set up an astronomical observatory and established a printing press. But in 1475 the Pope, Sixtus IV, invited him to Rome to take in hand the reformation of the calendar. Regiomontanus obeyed, but died shortly after his arrival—probably from the pest, although it was rumoured that he was poisoned by the son of George Trapezunt, against whom Regiomontanus, some years previously, had written a violent polemic.



NICHOLAS OF CUSA

The greatest published mathematical work of Regiomontanus is his *De triangulis omnimodis*, where trigonometry is first treated as an independent branch of mathematics. The treatise deals with both plane and spherical trigonometry, and it is worth noting that Regiomontanus was not familiar, at this time, with the notion of the tangent of an angle. Amongst the theorems on spherical trigonometry we find the statement that the three angles of a spherical triangle determine its sides, and the three sides determine the angles. It has been rightly remarked that the discovery of the first of these theorems must have presented,



at that time, special difficulties. The notion that size was not independent of shape must have been strange to one accustomed only to the properties of plane triangles. On several occasions in his writings Regiomontanus applies algebra to geometrical problems. We give an instance from the present work. The side bg and the height ad are given. The numbers given by Regiomontanus are 20 and 5 respectively. The ratio of the sides $ab : ag$ is given, viz. 3 : 5. It remains to determine ab and ag. Regiomontanus proceeds as follows: Since $ab < ag$, d lies nearer to b than to g. Make $de = bd$. Call $eg = 2x$. Then $be = bg - 2x = 20 - 2x$. Therefore $bd = 10 - x$, and its square is $100 + x^2 - 20x$. The square of ad is 25. Therefore $ab^2 = x^2 + 125 - 20x$. We also have that $dg = de + eg = 10 + x$. Hence

$$dg^2 = x^2 + 20x + 100, \quad ag^2 = x^2 + 125 + 20x.$$

From the given ratio we have

$$(x^2+125-20x):(x^2+125+20x)=9:25,$$

whence we obtain the quadratic

$$16x^2+2000=680x.$$

In this proof Regiomontanus uses the word *res* for x and *census* for x^2, and the signs + and − are not, of course, used by him. Although the notion of the tangent was not known to Regiomontanus at the time of writing this work, he later published a table of natural tangents, besides tables of sines. Other examples of his ability may be found from his letters, which include, amongst other things, indeterminate equations such as

$$17x+15=13y+11=10z+3.$$

Another example is

$$x+y+z=116,\ x^2+y^2+z^2=4624.$$

In judging the mathematical work of Regiomontanus we must remember that mathematics was a part only of his activities. He was an astronomer, an astrologer, and an inventor of mechanical devices. Also, his numerous travels must have made it difficult for him to give sustained attention to his work, so that he must have possessed a very formidable energy to have accomplished so much in his short life.

Minor Advances

From the time of Leonardo of Pisa no really comprehensive treatise on mathematics had appeared. It was not until the last years of the fifteenth century that another attempt was made to collect and expound existing mathematical knowledge. The attempt was only partially successful. The task was undertaken by a man of no great mathematical ability and whose chief qualification was the undoubted interest he took in the subject. In 1494 there was printed at Venice the *Summa de arithmetica, geometria, proporzioni e proporzionalità* of *Luca Paciuolo*, a Franciscan friar. The work appeared in very many subsequent editions.

Luca was employed by his order to lecture on mathematics, and wandered constantly about Italy, professing mathematics at Perugia, Rome, Naples, Venice, Milan, Florence, Bologna. If we compare the *Summa* with the *Liber Abaci* we find nothing of importance that is new. The *Summa* is based on Leonardo's work, but employs a better notation.

In the first part of the work Paciuolo expounds the rules of arithmetic. He gives no less than eight methods of performing multiplication. In the extraction of roots he gives a method of approximating to irrational values, called *surdi*. Suppose that $\sqrt{A}$ is irrational, and that a is an approximate value. Then

$$a + \frac{A - a^2}{2a} = a_1$$

is a second approximation,

$$a_1 + \frac{A - a_1^2}{2a_1} = a_2$$

is a third approximation, and so on. He also discusses Leonardo's favourite method, the rule of false assumption. In algebra he deals with both simple and quadratic equations. As illustrating the difficulties encountered by early mathematicians we may mention that Paciuolo is much puzzled by the equation $ax = bx$. If $a = b$ then we are given no information about x. If, on the other hand, a is not equal to b, the equation is impossible, for the greater cannot be equal to the less. It is interesting to note that the solution $x = 0$ did not occur to him. He gives equations of the type

$$ax^4 + cx^2 + e = 0$$

which are obviously reducible to quadratics, but he also gives two equations,

$$ax^4 + cx^2 = dx, \quad ax^4 + dx = cx^2,$$

against which he writes the word *Impossibile*. He considers these cubic equations impossible in the same sense that the quadrature of the circle is impossible, meaning evidently that no known

LUCA PACIUOLO

methods would solve them. The numerous examples that he gives, leading to simple or quadratic equations, often have more than a mathematical interest, for they throw a good deal of light on the conditions of commerce in the Italy of that time.

The geometrical part of the treatise contains nothing of great interest; it is worth noting, however, that Paciuolo, like Regiomontanus, uses algebra in the solution of geometrical problems.

His notation, as we have said, is an improvement on that of Leonardo. Addition is indicated by p or $\bar{p}$, and subtraction by $\bar{m}$ or by *de* for *demptus*. And equality is sometimes indicated by *ae*. The signs + and – and the practice, introduced by Jordanus, of representing quantities by letters, were not known to Paciuolo. But a work had appeared, by the German *Johannes Widmann*, published at Leipzig in 1489, in which the signs + and – had appeared. This is the first appearance of these signs of which we have any knowledge. Several theories of their origin have been put forth, but no explanation is entirely satisfactory.

Paciuolo was a friend of *Leonardo da Vinci*, and it has sometimes been claimed for the latter that he was a good mathematician. The evidence that we have does not support this view. He discusses no problem of great difficulty, and he sometimes accepts as true a solution which is, in fact, false. In the department of mathematics, at any rate, it appears that Leonardo's reputation rests chiefly on more or less vague surmises which he did not prove and was most probably incapable of proving. They are a sufficient testimony to the activity of his intelligence and curiosity, but they do not enable him to rank as a great mathematician. In mechanics he was much more successful.

An important name yet to be mentioned which belongs to this century is that of the Frenchman *Chuquet* whose *Triparty en la science des nombres* was written in 1484. Chuquet clearly recognized the use of both positive and negative integral exponents. He gave the 'rigle des nombres moyens' which consists in pointing out that $\frac{a_1 + a_2}{b_1 + b_2}$ lies between $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$. He used this rule to approximate to the value of an unknown quantity by taking two values, one too large and the other too small, and then repeatedly applying the rule. Chuquet also concerned himself with indeterminate equations. He was in advance of his time, particularly in France, which lagged behind Italy in the art of calculation, so that his influence was slight.

In surveying the period of three hundred years that separates the *Liber Abaci* from the end of the fifteenth century we see that comparatively little has been accomplished. Trigonometry, it is true, has been put into good shape by Regiomontanus, and algebra has been applied, in a limited manner, to geometrical problems. But algebra and geometry have undergone very little development. On the other hand, in the latter part of the fifteenth century, there were, scattered and disconnected, the beginnings of a good notation, in the absence of which the development of algebraic thought would have been yet further retarded.

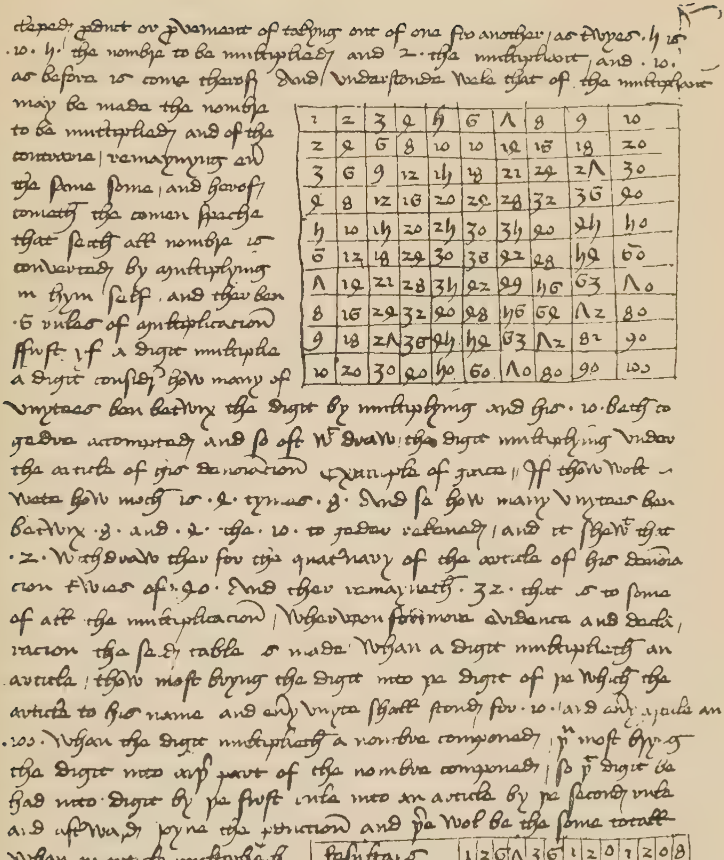
4

The Beginnings of Modern Algebra

Michael Stifel

During the sixteenth century the pace of mathematical discovery quickens and, in order to confine our attention to the most important advances, the work of many writers who would have required mention in any preceding century must be passed over. Notation continued to be improved, and in this connexion we must mention the algebra published by *Christoff Rudolff* in 1525 under the title of *Die Coss*. The sign √ occurs in this work to indicate the square root, √ √ √ indicates the cube root, and √ √ the fourth root. His contemporary, *Adam Riese* (1489–1559), who became a well-known teacher, published an arithmetic in 1536 in which the signs + and − are used.

These signs are also used in the work of *Michael Stifel* (1486–1567). But Stifel's work is interesting on other grounds. He has been described as the greatest German algebraist of the sixteenth century, and he was certainly one of the oddest personalities of the time. His best-known mathematical work is his *Arithmetica integra*, published at Nuremberg in 1544. The reforming divine Melanchthon, who took a meritorious part in helping the spread of mathematical learning by his prefaces and addresses, wrote a preface for this work. The work consists of three parts, of which the first deals with rational numbers, the second with irrational numbers, and the third with algebra. In the first part occur two very interesting sections ; in one Stifel emphasizes the great advantages that may be drawn from associating an arithmetical with a geometric progression—from which it has been inferred that he had glimmerings of the notion of logarithms—and in the second he



depart, parte or remanent of takyng out of one fro another, as thus 4 is 10. 4 the nombre to be multiplied and 2 the multipliant, and 10 as before is come therof. And understonde wele that of the multipliant may be made the nombre to be multiplied and of the contrarie, remaynyng ay the same somme, and herof cometh the comen practise that seith all nombre is conuertyd by multiplying in hym self. And ther ben 6 rules of multiplicacion. First if a digit multiplie a digit consider how many of unytees ben betwix the digit by multiplying and his 10 bethe to gedre acomptyd and so oft wt draw the digit multiplying under the article of his denominacion. Example of grace. If thou wolt wete how moch is 4 tymes 8. Se how many unytees ben betwix 8 and 4 the 10 to gedre rekened, and it shew that 2 withdrawe ther fro the quaternary of the article of his denominacion twies of 40 and ther remayneth 32 that is to somme of all the multiplicacion. Wherupon for more evidence and declaracion the seid table is made. Whan a digit multiplieth an article, thou most bryng the digit into ye digit of ye which the article to his name and euery unyte shall stond for 10 and euery article an 100. Whan the digit multiplieth a nombre componed, yu most bryng the digit into eueryche part of the nombre componed, so yt digit be had into digit by ye first rule into an article by ye second rule and afterward ioyne the product and ye wol be the somme totall. Whan an article multiplieth an article the digit wherof he is named is to be brought into the digit wherof the other is named and euery unyte wol be worth

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Resultantes	1 2 6 7 3 6 1 2 0 1 2 0 8
Multiplicandi	2 3 2 6 4
Multiplicantes	6 3 2 3 2 0 3 0 2

Page from a fifteenth-century translation into English (Bodleian Library, Ashmole 396) of the *Algorism* of John Holywood. The Arabic (i. e. Hindoo) numeral notation is shown.

gives the numbers which occur in a binomial expansion up to the seventeenth order. The rest of the treatise is less remarkable. Part two is essentially an algebraic translation of the tenth book of Euclid, while the third part deals with equations some of which are cubics which can be reduced by special devices to equations of a lower order. It is worth mentioning that Stifel defines negative, or, as he calls them, absurd, numbers to be numbers less than zero. In his solutions of equations he, like practically all the other mathematicians of this century, discarded negative roots. As we have said, he used the signs + and – ; he also employed the sign $\surd$ and, in one place, he represents a number of unknown quantities by letters.

An improved edition of Rudolff's *Die Coss* was brought out by Stifel in 1553. The appendix to this work, although of no importance to mathematics, is interesting for the light it throws on Stifel, who considered it quite the most important part of his work. Stifel was originally a monk, but was converted to the doctrines of Luther, of whom he was a personal friend. His conversion was brought about by his discovery that the number 666, the number of the beast in the Book of Revelation, really referred to Pope Leo X, who was head of the papacy from 1513 to 1521. He demonstrates his discovery in the following way: If Leo DeCIMVs be written, the letters MDCLVI make up the number 1656. This is too great by 990. It is clearly indicated, therefore, that the name should be followed by the symbol X. The result is now too great by 1000. But the 1000 springs from the letter M which, in this case, must stand for *Mysterium* and not for a number. Thus the required result is obtained. It is obvious from this example that Stifel had the flexibility in interpreting his material so necessary to the successful cabbalist, and he seems to have spent the rest of his life in perfecting this gift. The most interesting of his results was his discovery, reached by an analysis of biblical writings, that the world was to come to an end on 3 October 1533. The peasants of his district believed him and, as the date was near, abandoned all work, with the result that, when the day came and passed, many of them found themselves ruined. Stifel had to seek refuge in the prison at Wittenberg, from which he was released by Luther. The only point about these interpretations which is of mathematical interest is that they prove that Stifel was familiar with 'triangular numbers', i.e. numbers of the form $\frac{n(n+1)}{2}$; for, in giving numerical values to the letters of the alphabet, he gave them the values of these numbers, 1, 3, 6 ... up to 276.

Amongst the minor writers of this period we may mention *Robert Record* (1510–58) who, in an arithmetic called the *Grounde of Artes*, published in 1540, uses the signs + and −, and who, in an algebra called the *Whetstone of Witte*, published in 1557, uses the sign = for equality. This is the first time that this sign appears. Record says he adopted a 'pair of paralleles' for this sign 'because noe 2 thynges can be more equalle'.

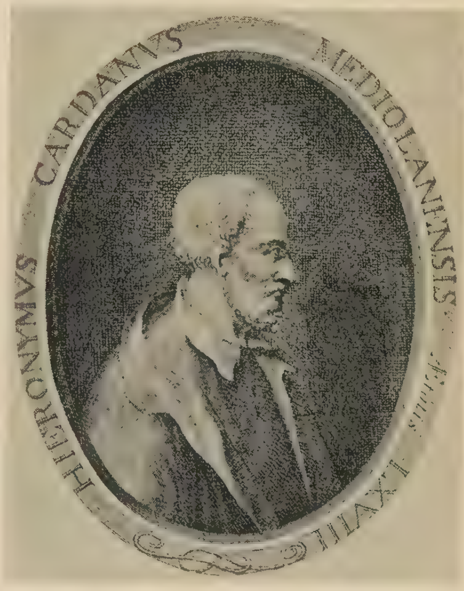
The Cubic Equation

We now come to consider one of the most interesting advances in the early history of mathematics. As we have seen, Paciuolo concluded his great treatise by giving two examples of cubic equations and declaring them to be impossible. The early half of the sixteenth century saw the solution. This great step forward in the creation of algebra is popularly attributed to Cardano. More thorough investigation seemed to show, however, that Cardano merely borrowed from Tartaglia, but there is reason to suppose that, after all, Cardano's reputation is largely justified, although it is certain that he was not the first European to solve a cubic equation. The detailed investigation of this historical question is lengthy and involved, but it is of sufficient interest and importance to warrant us in giving the main facts.

Four names are involved: *Scipione del Ferro*, *Hieronimo Cardano*, *Nicolo Tartaglia*, *Luigi Ferrari*. Del Ferro was professor at the university of Bologna and died in 1526. Amongst the papers he left behind, but which never came to be printed, was a solution of the equation $x^3 + ax = b$. What happened to these papers is not known, but one Florido, a pupil of Del Ferro's, became acquainted with the solution of this special form of the cubic. At what date Del Ferro made his discovery is not known; it is probable, however, that it was not later than 1515. In 1535 Florido, hearing that Tartaglia, professor at Venice, claimed to be able to solve cubic equations, challenged him to a contest. According to Tartaglia's own account, this challenge spurred him on to find a general rule for solving the equation $x^3 + ax = b$. He succeeded just eight days before the contest, and he also found the solution of $x^3 = ax + b$.

Ten years later appeared Cardano's famous treatise *Ars magna de rebus Algebraicis*, and in this treatise solutions of cubic equations are given. Cardano, while writing his treatise, had begged Tartaglia to communicate to him his solution of the cubic. Tartaglia had at first refused, but had yielded later, after first extracting a pledge of secrecy from Cardano. When, therefore, the *Ars magna* was published, Tartaglia angrily accused Cardano of a breach of faith. There followed a long and bitter controversy between Tartaglia and Cardano's pupil, Luigi Ferrari. In the course of this controversy Ferrari hints that Cardano had obtained his solution from a third person who had, like Florido, obtained the solution from del Ferro. This does not, however, affect the fact that Cardano had obtained a solution from Tartaglia. It has been suggested that Tartaglia, whom a close examination shows to have been a not very truthful person, did not really discover his solution, but had also, indirectly, derived it from del Ferro. But the discussion of this point is, in the absence of evidence, not very profitable, and it is safer to say that Tartaglia must share with del Ferro the honour of furnishing the first solutions of the cubic obtained by European mathematicians.

When we leave the question of priority, however, and consider the mathematical power shown in dealing with the new discovery, there is little doubt that the chief place must be accorded to Cardano. He was, indeed, one of the most gifted men of his time. He was physician, astrologer, physicist, mathematician, and philosopher, and his passionate and turbulent life showed that even these activities were not sufficient to absorb his formidable energies. In algebra he was the first to discuss negative and even imaginary roots. He knew that a cubic equation possessed three roots, and that the sum of the roots was equal to the coefficient of the quadratic term in the equation. He invented a method for approximating to a root of equations of higher order, and was acquainted with 'Descartes rule of signs'. He also solved correctly certain questions in probability. These achievements are very considerable, and there is nothing in Tartaglia's later work which reaches so high a level. Cardano's pupil, Ferrari, seems also to have been possessed of unusual ability, although we are chiefly indebted for our knowledge of his work to Cardano's reports,



since Ferrari published nothing independently. His best performance is his solution of the biquadratic equation which is minus the cubic term. The solution was invented to deal with the equation set by *Da Coi*;

$$x^4+6x^2+36=60x,$$

but the method is general. In describing it, therefore, we may consider the equation

$$x^4+ax^2+c=bx. \quad \text{........................} \quad (1)$$

Ferrari first makes the left-hand side a perfect square by adding $(2\sqrt{c}-a)x^2$ to both sides, obtaining

$$(x^2+\sqrt{c})^2=(2\sqrt{c}-a)x^2+bx. \quad \text{..................} \quad (2)$$

Now consider the identity

$$(A+B+C)^2=(A+B)^2+2AC+2BC+C^2$$

and in equation (2) regard

$$(x^2+\sqrt{c})^2 \text{ as } (A+B)^2, \text{ i.e. } x^2=A\,; \quad \sqrt{c}=B.$$

Choose a new variable t as being C in the identity. Then the left side of (2) is still a perfect square if we add to both sides of the equation

$$2AC+2BC+C^2=2x^2t+2\sqrt{c}\,.\,t+t^2,$$

the equation becoming

$$(x^2+\sqrt{c}+t)^2=(2\sqrt{c}-a+2t)\,x^2+bx+(t^2+2t\sqrt{c}).$$

If the right side of this equation is also a perfect square, then, on taking square roots, we have a quadratic equation in x. But the condition that the right side should be a square is that

$$4(2\sqrt{c}-a+2t)\,(t^2+2t\sqrt{c})=b^2$$

or

$$t^3+\left(3\sqrt{c}-\frac{a}{2}\right)t^2+(2c-a\sqrt{c})\,t=\frac{b^2}{8}.$$

The solution is thus made to depend on the solution of a cubic. This solution was published in the *Ars magna* when Ferrari was only twenty-three years of age.

Four years before Cardano's death *Rafaello Bombelli* published, in 1572, an algebra which carried the investigation of cubic equations a little further. In particular, he threw light upon complex roots. He improved the current notation in one or two points. Thus he represents the unknown quantity by ^{1}, its square by ^{2}, its cube by ^{3}, and so on. He also improved the notation for representing the extraction of roots. Paciuolo, for example, in extracting roots of compound quantities, used the *Radix universalis*, denoted by *℞V*, the symbol ℞, taken by itself, meaning the square root. Thus the expression $\sqrt{7+\sqrt{14}}$ would be written by Paciuolo *℞ V* 7 $\bar{p}$ *℞* 14. Bombelli, in taking the root of a compound expression, used the symbol $\llcorner$ at the beginning of the expression and the symbol $\lrcorner$ at the end, so that the expression was bracketed. Thus, in Bombelli's notation $\sqrt{7+\sqrt{14}}$ would be written *℞* $\llcorner$ 7 *p* *℞* 14 $\lrcorner$. As indicating the comparative obscurity of even this improved notation we give two specimens which occur in Bombelli's work.

$$℞\, q \llcorner ℞\, c \llcorner ℞\, q\, 68\, p\, 2 \lrcorner\, m\, ℞\, c \llcorner ℞\, q\, 68\, m\, 2 \lrcorner \lrcorner$$
$$= \sqrt{[\sqrt[3]{(\sqrt{68}+2)} - \sqrt[3]{(\sqrt{68}-2)}]}$$

$$℞\, c \llcorner 4\, p\, di\, m\, ℞\, q\, 11 \lrcorner\, p\, ℞\, c \llcorner 4\, m\, di\, m\, ℞\, q\, 11 \lrcorner$$
$$= \sqrt[3]{(4+\sqrt{-11})} + \sqrt[3]{(4-\sqrt{-11})}$$

It will be apparent from these specimens that the algebraical notation of this time must have obscured many simple mathemetical relations. It is, indeed, an astonishing fact that algebraic notation improved so slowly, seeing the quality of the mathematical work that was now being done.

Franciscus Vieta

In the sixteenth century, as we have seen, great advances in mathematics, particularly in algebra, were made. From the point of view of the mathematician the century is, indeed, predominantly algebraical, and we now come to consider the work of the greatest algebraist of this century. *Franciscus Vieta* (*François Viète*) was born in 1540 at Fontenay-le-Comte in

Poitou, and died at Paris in 1603. He first studied law, but gave up this occupation in 1567, and later became a member of the parliament in Rennes. In 1580 he was attached to the parliament at Paris as *Master of Requests*, and was thenceforward advanced to further offices till his death. He was not, therefore, a professional mathematician. He spent a great deal of time at the study, however, sometimes remaining for days together shut up in his room and hardly interrupting his work either for meals or sleep. And it is probable that, after 1580, he spent most of his time at mathematics. We have a record of one occasion when his extraordinary analytical ability was of service to the state. Henry IV had been much impressed by the way in which Vieta had solved a certain problem and, during the war with Spain, asked him to decipher a Spanish dispatch which had been intercepted. The Spaniards wrote their dispatches in a cipher containing more than 500 characters, and were perfectly confident that it was insoluble without the key. Vieta's analysis enabled him to find the key, however, and the French used it, to their own great profit, for the next two years.

Vieta's mathematical writings were numerous. He had them printed at his own expense, and sent copies to other mathematicians in all countries. Amongst his algebraical works we must consider first the *In artem analyticam isagoge*, published in 1591. In this work Vieta uses the signs + and − to denote addition and subtraction. He also uses the sign =, not, however, to denote equality, but to indicate the difference between two quantities when we do not know which is the greater. But a more important innovation is Vieta's use of letters to represent both known and unknown quantities. He used consonants to represent known quantities, and vowels to represent unknown quantities. Also, instead of denoting the powers of a quantity by separate letters, thus taxing the memory and obscuring relations, he used terms such as A quadratus, A cubus, &c., to denote x^2, x^3, &c. Thus the expression $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$ was written by him a cubus $+ b$ in a quadr. $3 + a$ in b quadr. $3 + b$ cubo aequalia $\overline{a+b}$ cubo. Vieta also laid stress on the homogeneity of an equation, attributing to the constants such dimensions as should make every term of the same dimensions. Thus he would write

A cubo $- D$ in A quadr $+ C$ plano in A aequatur Z solido.

In the *De aequationum recognitione et emendatione*, first published in 1615, twelve years after his death, Vieta occupies himself mostly with the theory of equations. In this treatise occurs some of his best work. We give, as an instance, his rapid solution of the cubic of the form $x^3 + 3ax = 2b$. Put $y^2 + xy = a$. Then $x = \frac{a - y^2}{y}$ and the original equation becomes $y^6 + 2by^3 = a^3$, a quadratic in y^3, from which x is immediately obtained.

His method of solving the biquadratic which has been deprived of its cubic term is similar to Ferrari's. Writing $x^4 + ax^2 + bx = c$ in the form $x^4 = c - ax^2 - bx$, and adding $x^2y^2 + \frac{1}{4}y^4$ to both sides of the equation, he obtains

$$(x^2 + \tfrac{1}{2}y^2)^2 = (y^2 - a)x^2 - bx + (\tfrac{1}{4}y^4 + c).$$

The condition that the right side of the equation is a perfect square gives

$$y^6 - ay^4 + 4cy^2 = 4ac + b^2,$$

a cubic in y^2.

Vieta, like nearly all other mathematicians of his century, did not recognize negative roots, and this fact hindered him, of course, in his researches on the theory of equations. But he thoroughly understood the relations between the positive roots of an equation and its coefficients. In the present treatise he expresses these relations for equations of the second, third, fourth, and fifth degrees. He states that $x = a$, $x = b$ are the two roots of

$$(a+b)x - x^2 = ab;$$

$x = a$, $x = b$, $x = c$ are the three roots of

$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x = abc$$

and so on.

In Vieta's last work, *De numerosa potestatum purarum atque adfectarum ad exegesin resolutione*, a method is given for approximating to a positive root of numerical equations. An example will make the method clear. In the equation $x^2+cx=a$ let x_1 be a known approximate value of the root, so that the root may be written as x_1+x_2. Substituting in the equation, we get

$$a=x_1^2+cx_1+(2x_1+c)x_2+x_2^2,$$

or

$$\frac{a-x_1^2-cx_1}{2x_1+c}=x_2$$

if we consider x_2 so small that its square may be neglected. Taking now the new approximate value x_1+x_2 we find, in the same way, another small quantity x_3. We give an example from Vieta.

$$x^2+7x=60750.$$

A first approximation is $x_1=200$. Then

$$\frac{60750-x_1^2-7x_1}{2x_1+7}=40 \text{ approximately.}$$

Taking 240 as the next approximation we get 3 for x_3, and the value 243 turns out to be exact. The way in which the method is to be modified for equations of different orders is easily seen. The calculations become more laborious the higher the order of the equation. Vieta himself gives some laborious examples, as in the example $x^6+6000x=191246976$.

Vieta was also a good geometrician, but his attempts to determine the value of π and his trigonometrical work are of greater interest. Using the method of Archimedes, Vieta found a value for π which is correct to nine decimal places. He stated, in fact, that π was less than 3·1415926537 and greater than 3·1415926535. But of greater interest than this is the fact that he obtained the value of π as an infinite product—the first infinite product to appear in mathematics. His result is

$$\frac{2}{\pi}=\sqrt{\tfrac{1}{2}}\cdot\sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\right)}\cdot\sqrt{\left\{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\right)}\right\}}\ldots \text{ ad inf.}$$

This product, as it happens, is convergent.

We have already said that Henry IV was much struck by the ability shown by Vieta in solving a certain problem. The problem had been set as a challenge to the mathematicians of the world by the Belgian mathematician *Van Roomen*, and consisted of an equation of the forty-fifth degree. The Belgian ambassador acquainted Henry IV with this problem, and the king sent for Vieta. In a few minutes Vieta had found two roots of the equation. The equation was

$$45x-3795x^3+95634x^5-1138500x^7+\ldots+945x^{41}-45x^{43}+x^{45}=B.$$

Vieta saw at once that this equation was simply the expression by which $B=2\sin\phi$ was given in terms of $x=2\sin\frac{\phi}{45}$, for he had previously worked out the expression of $\sin n\theta$ in terms of $\sin\theta$ and $\cos\theta$. Since Van Roomen constructed the equation, it follows that he also was familiar with this development. But he apparently did not realize, as Vieta did, that the expression, considered as an algebraical equation, has many roots. Vieta succeeded in finding 23 of the 45 roots. The remaining roots were, with his notions of negative quantities, unintelligible to him.

Before leaving this century mention must be made of *Simon Stevin* (1548–1620), who was one of three or four independent inventors of decimals. Stevin was firmly convinced of the great importance of the decimal system, and advocated it for money and weights and measures. His notation was rather clumsy. For instance, he would write 237·578 as 237⓪ 5① 7② 8③. But Stevin's great contributions were to the still young and feeble science of mechanics, and we shall treat of this part of his work in a separate section.

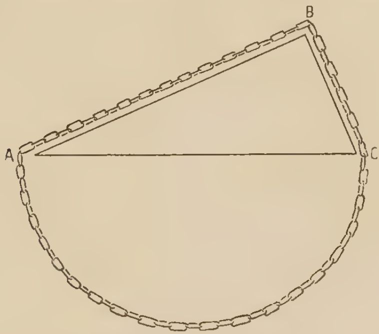
Up to the close of the sixteenth century we see that the chief mathematical developments, from the time of Leonardo of Pisa, have been algebraical. Trigonometry, both plane and spherical, has also been developed, but nothing of real importance has been done in geometry. It is not until we reach the time of *Descartes* that geometry undergoes any great development. So far Italy has played the leading role. Germany has made valuable contributions, particularly in algebra, and France, until the appearance of Vieta, has, like England, played a subordinate role. During the seventeenth century some of the very greatest names in the whole history of mathematics appear, and almost every country in Europe cultivates mathematics with energy.

5

Related Advances

Mechanics

By the end of the sixteenth century, as we have seen, fairly clear notions of many of the primary entities of pure mathematics had been obtained. The notions of point, straight line, curved line, plane, solid, as defined by the Greeks, were part of the ordinary mental furniture of mathematicians. These concepts had not attained the rigour which has since been given to them, but they were fairly clear, and by using them mathematicians were led to accurate results. The notion of number, also, had been extended to include the negative, the irrational and the imaginary. These notions were in an elementary stage, but they existed and were receiving more and more attention. The conceptions belonging to *mechanics*, however, were in a very different case. At this time it was generally held, for instance, that the motion of a body required some sort of action to maintain it. Motion was produced by an impulse, and on the cessation of the impulse the motion also should cease. The fact that a cannon ball continued to move after the explosion of the powder was attributed to the reaction of the medium in which it was moving. The medium, in some way, helped the cannon ball along. Another belief was that a body cannot have two motions at the same time. A cannon ball shot off in a horizontal line could not be simultaneously falling towards the earth. It was believed that the cannon ball persevered in a straight line and then suddenly dropped, though Leonardo da Vinci had long ago taken the correct view. Statical notions were just as confused. The floating of solids in a liquid, for instance, was supposed to depend only on their form. And we may mention the well-known but almost incredible fact, that, following Aristotle, heavy bodies were

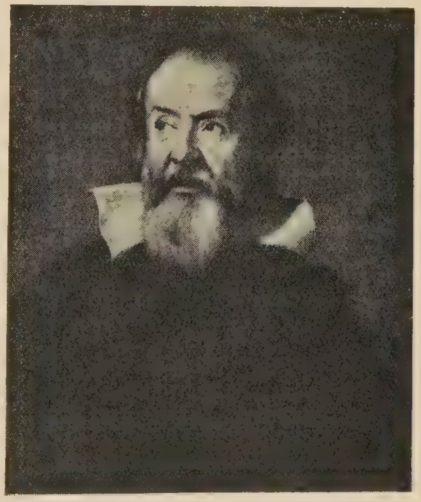


believed to fall faster than light ones, with a velocity proportional to the weight. It is extraordinary that the simple experiment of dropping stones of different weights simultaneously should have required the genius of Galileo to hit upon it.

The first man to clear up some of this confusion was Simon Stevin, whose introduction of decimals we have already referred to. His *Statics and Hydrostatics*, published in Flemish at Leyden in 1586, uses as its fundamental theorem the triangle of forces which becomes, by an obvious extension, the parallelogram of forces. The manner in which he discusses the problem of a weight resting on an inclined plane, a problem much discussed at that time, has often been quoted. In the triangle *ABC* Stevin imagines a chain of weights so arranged that the total weights supported by the sides *AB* and *BC* are proportional to the lengths of the sides. Now experience teaches us, says Stevin, that such a chain of weights will remain immobile. For if it moved we would have perpetual motion, which is absurd. But it is obvious that the equilibrium will be unaffected if the lower part of the chain, from *A* to *C*, be removed. Accordingly the weight on *BC* balances the weight on *AB*. And this would remain true even if *BC* were vertical. We thus reach the theorem that the component of *W*, the weight of *AB*, in the direction *BA*, is *W* sin *CAB*. Stevin also discussed questions in hydrostatics, explaining the so-called hydrostatic paradox, and investigating the equilibrium conditions for loaded ships.

The fundamental notions of dynamics were made considerably clearer by *Galileo Galilei* (1564–1642). Indeed, of all the work performed by this energetic and versatile genius, perhaps his work on mechanics should take the first place, although it is much less dramatic, of course, than his discoveries in observational astronomy. While still a student at Pisa Galilei noticed that the period ot oscillation of a swinging candelabrum in the cathedral was the same whether it was describing large or small arcs. He timed the decreasing swings by his pulse. He made experiments, and demonstrated that the period of a pendulum is also independent of the weight of the pendulum. Whether the pendulum be light or heavy, or describing either large or small arcs, the period of oscillation is the same if the length is the same. His famous experiment of dropping bodies of different weights from the leaning tower of Pisa enabled him to demonstrate that all bodies undergo the same acceleration in falling towards the earth, a result which his experiments with light and heavy pendulums also proved.

The bulk of Galilei's dynamical ideas are contained in his *Discorsi e dimostrazioni matematichè intorno à due nuove scienze*, published at Leyden in 1638. In this volume he attempts a disproof of the Aristotelian doctrine, that the velocities of falling bodies are proportional to their weights, on *a priori* grounds. Thus, he says, if a body of weight 10 falls ten times as fast as a body of weight 1, a combination of these two bodies should fall with some intermediate velocity, since the slower velocity of the weight 1 will retard the velocity of the weight 10, that is, a body of weight 11 should fall more slowly than a body of weight 10, which contradicts the assumption. In his investigation of the form assumed by a heavy chain suspended at its two ends, Galilei concludes that it is a parabola—a result which is definitely wrong. Two theorems are enunciated respecting uniformly accelerated motion. The first is that the time taken to describe a straight line by a uniformly accelerated body which starts from rest, is the same time as would be required to cover the same distance by a body moving with a uniform velocity which is one-half the final velocity reached by the uniformly accelerated body. The second theorem states that the distance covered by a uniformly accelerated body is proportional to the square of the time taken in covering the distance. How difficult the formation of correct dynamical ideas was at that time, however, is shown by the fact that even the insight of a Galilei could be led sadly astray. Thus, in Galilei's dialogues on the system of the world he explains the tides by saying that they are caused by the different velocities



GALILEI

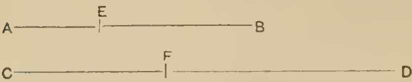
of rotation possessed by different parts of the earth. It is possible that he was led to this hypothesis by his desire to reject Kepler's notion that the attraction of the moon caused the tides, for Galilei does not seem to have shown a welcoming spirit towards Kepler's discoveries. Thus Kepler's three laws of planetary motion, an astronomical advance of the first importance, are completely ignored by Galilei. On another point which had been made an objection to the Copernican hypothesis, viz. the objection that a stone thrown up in a straight line should not, if the earth is moving, fall down on the same spot that it left, Galilei's reply is in accordance with sound dynamical principles. Galilei also introduced elementary and clear ideas into statics and hydrostatics, and also proved that the path of a projectile is a parabola. But although Newton's first two laws of motion and possibly also the third law can be deduced from Galilei's work, he nowhere enunciates them distinctly. A perfectly clear and complete system of dynamics was not obtained until the time of Newton.

We must refer also to the work of *Evangelista Torricelli* (1608–47) who was associated with Galilei for the last few years of the latter's life. Torricelli was present when Galilei was requested by some members of a Florentine guild to alter some pumps so as to make them raise water through more than thirty feet. Galilei suggested they should first find out why the water rose at all. When they explained that it was because nature abhorred a vacuum, Galilei pointed out that the interesting point was that nature did not abhor a vacuum of more than thirty feet. Torricelli, his attention thus aroused, worked at this problem, and finally constructed the first mercury barometer. He also enunciated correct theorems in hydrodynamics.

Logarithms

Early in the seventeenth century arithmetical calculations promised to be a curse. The new ideas in astronomy, and the new instruments, required, for their proper utilization, new tables of all kinds. The calculation of these tables required years of labour. Some of the mathematicians of that time, it is true, seemed to take a delight in arithmetical drudgery. Thus even Vieta proposed to himself entirely useless problems, apparently for the pleasure of spending days or weeks in arithmetical calculations. *Van Ceulen*, a contemporary of Vieta's, seems to have devoted all his energies to obtaining closer and closer approximations to the value of π. His most thorough-going calculation gives the value of π correct to 35 places of decimals. But even if such devoted souls are bound to appear in every generation it is greatly to the advantage of science that their energies should be more economically employed. We can therefore understand the enthusiasm which greeted Napier's invention of logarithms. It appears that he had a precursor in *Joost Bürgi*, a Swiss. But Bürgi's tables were crude, and he did not publish them until some time after Napier's work had appeared. In estimating Napier's achievement we must remember that the method of representing powers by indices was not known, and his invention was, accordingly, the result of a highly original and powerful mental effort.

John Napier (1550–1617) Baron of Merchiston, in Scotland, published his *Mirifici logarithmorum canonis descriptio* in 1614 in which, besides explaining his logarithms, he gives a table of logarithms of the natural sines from 0° to 90° in steps of a minute. Napier obtains the notion of a logarithm by comparing two motions, as follows: Let a particle describe the line AB in such a way that its velocity at any point E is proportional to the distance EB that it has still to traverse. Let another particle



start from C on the line of indefinite length CD with the same initial velocity as the particle on AB, and let it preserve this velocity unchanged. Then, when the particle on AB is at E, the particle on CD will be at some point F. Napier calls CF the logarithm of BE. A simple calculation shows that these are not what are known as natural logarithms. They decrease as the number increases. In his tables Napier takes the 'whole sine', i.e. $\sin 90° = 10^7$ and puts zero as its logarithm. Then the logarithm increases as the angle decreases from 90°.

JOHN NAPIER

Henry Briggs (1556–1631), at that time professor at Gresham College, London, and afterwards first Savilian professor at Oxford, was very greatly struck by Napier's work, and journeyed to Scotland to see him. He suggested to Napier that the logarithms would be more convenient if 10^{10} was chosen as the logarithm of the tenth part of the whole sine, leaving zero as the logarithm of the whole sine. Napier had already thought of this, and further suggested that zero should be taken as the logarithm of unity, and 10^{10} as the logarithm of the whole sine. This makes the characteristics of numbers greater than unity positive. The labour of calculation was undertaken by Briggs, and in 1624 appeared his *Arithmetica logarithmica* giving logarithms, to fourteen places, for numbers 1 to 20,000, and for numbers 90,000 to 100,000. Logarithms immediately became popular all over Europe, partly owing to the influence of Kepler, who constructed tables of his own. Briggs tables were completed by a Dutchman, *Adrian Vlacq*, who published, in 1628, a table of logarithms, to ten places of decimals, for all numbers from 1 to 100,000, using Briggs's logarithms, to ten places, for the numbers calculated by Briggs.

Notation

Algebraic notation at this time was further advanced by the writings of other English mathematicians. *Thomas Harriot* (1560–1621) was one of the leading writers of his time on the theory of equations. He only recognized positive roots, however, and even attempted to prove that none but positive roots could exist. He introduced the signs $>$, $<$ to represent greater than, less than, and he improved the notation for representing powers of the unknown quantity. Thus he wrote the square of a, as aa, its cube as aaa, &c. His contemporary, *William Oughtred* (1575–1660), in his text-book on arithmetic, *Clavis Mathematicae*, published in 1631, introduced the sign $\times$ for multiplication. He also introduced the symbol : : in proportion. In a work on trigonometry, published in 1657, he uses abbreviations for sine, cosine, &c. These latter innovations were subsequently forgotten however.

6

New Methods

The Method of Indivisibles

In considering the mathematical work of *Johann Kepler* (1571–1630) one is chiefly struck by the quality of his imagination. In his astronomical work this quality is perhaps even more clearly perceived. As compared with, for instance, Galilei, Kepler's imagination ranges farther and is less hindered by considerations of probability. Kepler was, apparently, able to believe any superstition and seriously to consider any hypothesis, however extravagant. His chief hindrance in effecting his great scientific discoveries seems to have been the difficulty he experienced in hewing a path through the jungle of his own fancies. But combined with this luxuriant imagination was a very delicate insight. In the course of these wild flights Kepler could be trusted, sooner or later, to hit upon the recondite and subtle truth of the matter and, having once reached it, to hold it fast. An account of Kepler's astronomical speculations is outside our scope, but we may refer here to his enunciation of what is now called the principle of continuity in geometry. It appeared in his commentary on the work of Witelo, a Polish mathematician of the thirteenth century, *Ad Vitellionem paralipomena, quibus Astronomiae pars optica traditur*, published in 1604. He considered the conic sections to form five species, from the line-pair to the circle. From the line-pair we may reach the parabola by passing through an infinity of hyperbolas and, having reached the parabola, we may pass through an infinity of ellipses to the circle. To make this process clear Kepler gave the name of *foci* to certain remarkable points connected with these curves. The circle has one focus at the centre, the ellipse and the hyperbola two foci equidistant from the centre. The parabola also has two foci, one within it and the other at infinity on the axis, within or without the curve. The line-pair must also be regarded as possessing two foci but, as in the circle, they coalesce and, further, fall upon the lines themselves. Kepler defends the use of 'however absurd expressions' if the analogies they suggest make the essence of the thing clear to us.

This generalized view of conics is a very good specimen of Kepler's insight. The whole point of view has turned out to be of the greatest importance in mathematics. It initiated one of the rare and valuable departures in mathematics which result from an effort of imagination rather than from an effort of ratiocination. The same faculty is exhibited in Kepler's method of evaluating areas and volumes, a method closely akin to that of the integral calculus. The year 1612 was, in Austria, an exceptionally rich wine-year, and Kepler, on purchasing his wine, was astonished at the crude methods adopted for evaluating the volume of the barrels. He reflected on the matter, and the result of his reflections was the *Stereometria doliorum*, published in 1615. In this work he discusses the solids known to Archimedes and also some new ones, and then deals with the problems presented by the wine barrels. His method is to regard an area or volume as made up of an infinite number of parts. Thus he regards the circumference of a circle as a polygon possessing an infinite number of sides. Each of these sides is taken as the base of a triangle whose apex is the centre of the circle, and the area of the circle is taken to be the sum of the areas of these triangles. Similarly,

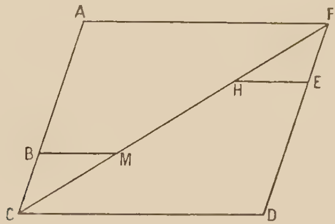
JOHANN KEPLER

the volume of a sphere is built up from an infinite number of cones. He also discusses solids generated by revolving a curve about an axis, and in the last section deals with problems of maxima and minima. These investigations were preceded by a calculation in his *Astronomia nova* of 1609 in which Kepler solves the problem we should now describe as evaluating the integral $\int_0^\phi \sin\phi \,.\, d\phi = 1 - \cos\phi$. In the same work he also states that the circumference of the ellipse of axes $2a$, $2b$ is very approximately $\pi(a+b)$. It is not too much to say of Kepler that, besides originating the doctrine of geometrical continuity, he also originated, although in a crude and imperfect form, the infinitesimal calculus.

This part of his work was carried a stage further by *Bonaventura Cavalieri* (1598–1647), professor of mathematics at Bologna from 1629 till his death. Cavalieri published his *Geometria indivisibilibus continuorum nova[1] quadam ratione promota* in its first form in 1635, and the final edition, embodying his corrections, was not issued until 1653. In the meantime, in 1647, his *Exercitationes geometricae sex* had appeared. Cavalieri was familiar with Kepler's *Doliorum*, possibly through Galilei, and there can be little doubt that his method of indivisibles was suggested by that work. Although Cavalieri prefaces his work with a large number of definitions, what precisely he understands by his indivisibles is not very clear. He regards a line as made up of points possessing no magnitude, a surface as made up of lines without breadth, and a volume as made up of surfaces without thickness. But he also regards a surface as being generated by a moving line, and a volume by a moving plane.

A simple example of his actual method is given by his proof that a parallelogram is divided by its diagonal into two triangles each having half the area of the parallelogram. In the figure take $FE = CB$ and draw the lines HE, MB. The triangle FCD is considered as built up of lines HE, and the triangle CAF of lines BM. The lines HE, BM are equal, and such pairs fill up the whole of both triangles. Hence the totality of lines forming one triangle is equal to the totality of lines forming the other. Hence the triangles are equal and hence each is half the parallelogram.

Cavalieri also sums the squares and higher powers of such lines. Thus he states that the sum of the squares of the lines constituting one of the triangles above is equal to one-third the sum of the



squares of the lines constituting the parallelogram. This is equivalent to saying that

$$\int_0^a \frac{kb^2x^2}{a^2}\,dx = \frac{1}{3}\int_0^a kb^2\,dx,$$

which is correct. Cavalieri's own proof is cumbersome. By considering a number of special cases Cavalieri finally arrived at a theorem which is equivalent to saying that

$$\int_0^a \frac{kb^n x^n}{a^n}\,dx = \frac{kab^n}{n+1}.$$

Objections were made to Cavalieri's conception of indivisibles, particularly by *Guldin*, and Cavalieri, to meet these objections, recast his statements, but without succeeding in giving to them a satisfactory logical basis.

Very much the same method of indivisibles was hit upon, probably independently, by *Giles Persone de Roberval* (1602–75), professor of mathematics at Paris. But he differs from Cavalieri in regarding lines as made up of infinitely small lines, surfaces of infinitely small surfaces, and volumes of infinitely small volumes.

New Methods in Geometry

Kepler's geometrical ideas bore fruit in the work of *Gérard Desargues* (1593–1662), a geometrican of great ability and originality. He was not a professional mathematician, being an engineer and architect, but he came into contact with the greatest mathematicians of his time, two of whom, Descartes and Pascal, were able to understand him and appreciated him highly. But the invention of analytical geometry turned men's attention from the new paths opened up by Desargues, and it was not until recent times that his methods were thoroughly mastered, and a new branch of geometry, Descriptive Geometry, thoroughly established.

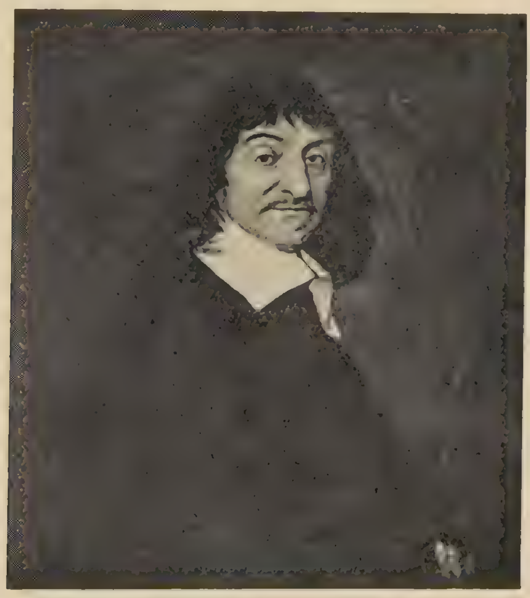
It was in 1639 that Desargues' work on conics appeared under the title *Brouillon project d'une atteinte aux événemens des rencontres d'un cone avec un plan.* In this work he states that parallel lines may be regarded as meeting in a point at infinity, and parallel planes in a line at infinity. The theory of involution of six points is discovered and discussed in detail. He introduces the method of passing from the properties of the circle to the properties of other conic sections by perspective. He also understands the theory of poles and polars. These general methods are illustrated by several beautiful theorems, some of which are amongst the most important that have been obtained in this department of geometry. Important as this work was, it was appreciated by very few mathematicians, and copies of Desargues' work became extremely rare. The new methods invented by Descartes and Fermat, on the other hand, excited much attention, although it was not until the infinitesimal calculus had been developed by Newton and Leibniz that the immense power of co-ordinate geometry was fully revealed.

The middle years of the seventeenth century constitute the greatest period of mathematical activity that we have yet encountered, and in describing the work of this period it must be remembered that mathematicians no longer worked in comparative isolation. The results achieved by such men as Descartes, Pascal, Fermat, were known by one another as soon as they were published, or even before, and a rivalry, always fruitful if not always friendly, continually incited them to fresh efforts.

René Descartes, usually regarded as the inventor of co-ordinate geometry, was born near Tours in 1596 and died at Stockholm in 1650. It was in 1628 that he moved to Holland, then at the height of its power, and for the next twenty years devoted himself to meditating on philosophy and mathematics. He was well adapted to live such a life, for he had great intellectual energy and ability, was unmarried, and possessed a cold, prudent, and selfish disposition which effectively protected him from all distractions. With his philosophy we are not concerned, but one of the three appendices to his famous *Discours de la méthode* is called *La Géométrie*, and this work contains the first exposition of analytical geometry. Its date of publication is probably 1638. Another work which we must take into account is his *Principia Philosophiae*, published in 1644, which contains his laws of motion and theory of vortices. An earlier work, *Le Monde*, written between 1629 and 1633 is also concerned with physical science, but was abandoned and left incomplete when Descartes learned that it would probably antagonize the church.

La Géométrie is obscure. It is by no means a systematic exposition of the new method ; the reader has very largely to reconstruct the method for himself. Part at least of this obscurity is intentional. 'I have omitted nothing', says Descartes, 'except by design. I had foreseen that certain people who vaunt that they know everything, would not have failed to say that they knew already all that I had written, had I made myself more intelligible.' As a result of this precaution the book was too difficult to be widely read, and it was not until 1659, when a Latin translation with explanatory notes by F. de Beaune, and a commentary by F. van Schooten, was published, that it became generally known. The book is divided into three parts; the first two treat of analytical geometry, and the third of algebra.

Algebra had already been applied to geometry by other writers, as we have seen. The wholly new contribution made by Descartes was in importing the idea of *motion* into geometry. It is said that the idea came to him while lying in bed and watching the movements of a fly crawling near an angle of the room. He saw that its position at any moment could be defined by its perpendicular distance from the ceiling and two adjacent walls. Thus he saw a curve as described by a moving point, the point being the point of intersection of two moving lines which were always parallel to two fixed lines at right angles. As the moving point described the curve, its distances from the two fixed axes would vary in a manner characteristic of the curve, and an equation between these distances could be formed which would express some property of the curve. Algebraical transformations of this equation would then reveal other properties of the curve. Similarly, an equation between two variables would represent some curve. Two curves could be regarded as lying together in the same plane, and then a consideration of their equations would determine what pairs of values their variables had in common, i.e. at what points they intersected. In the first part of *La Géométrie* Descartes discusses the problem which had led him to invent co-ordinate geometry. The problem is due to Pappus, and may be stated: 'Given several straight lines in a plane, to find the locus of a point such that the perpendiculars, or, more generally, straight lines at given angles, drawn from the point to the given lines, shall satisfy the condition that the product of certain of them shall be in a given ratio to the product of the rest.' In the second part Descartes distinguished between geometrical and mechanical curves, or, as Newton called them, algebraical and transcendental curves, and confined his attention



RENÉ DESCARTES

to geometrical curves. An important part of Descartes' work is his method of constructing tangents. He first found the circle which cut the curve in two consecutive points, and then drew the tangent to the circle. Descartes himself was extraordinarily pleased with this device, although it is by no means the most direct method of constructing a tangent. In the algebraic portion of his work Descartes made the important innovation of using exponents systematically. He also introduced the custom of denoting constants by the first, and variables by the last, letters of the alphabet. And he gave a rule for determining the number of positive and negative roots in an equation, which is accurate if the equation has no imaginary roots. It appears, however, that Descartes understood imaginary quantities, and he certainly fully understood negative quantities. The method of indeterminate coefficients is also due to him, although he mistakenly believed that it was a method by which algebraic equations of any order could be solved.

In the *Principia* Descartes expounds his celebrated theory of vortices. The foundations of the theory are metaphysical and lead to ten laws of nature, of which the first two are practically the same as those given by Newton, and the rest are inaccurate. It is not necessary to describe the theory since, in a deadly analysis in the second book of his *Principia*, Newton showed that the consequences of the theory are (1) inconsistent with the observed facts of planetary motion as described in Kepler's laws, (2) inconsistent with the fundamental laws of mechanics, and (3) inconsistent with the fundamental laws of nature assumed by Descartes himself.

The first half of the seventeenth century, therefore, saw the origination of three new and important branches of mathematics; the infinitesimal calculus, projective geometry, and co-ordinate geometry. Of these the first and third were certainly the most important, and it was partly because mathematicians realized this fact that projective geometry was comparatively neglected.

7

The Middle Seventeenth Century

Fermat, Pascal, Wallis

AMONGST Descartes' contemporaries the one who did most to extend mathematics was *Pierre de Fermat* (1601–65). It is possible that Pascal had even greater mathematical ability than Fermat, but he accomplished comparatively little, and there was no mathematician of the period who occupied himself with more varied problems than did Fermat. He published very little in his lifetime, and many of his discoveries exist in the form of marginal jottings. Such information as we have about the dates of his discoveries is obtained chiefly from his correspondence. It thus appears that he had discovered the principles of analytical geometry for himself some years before Descartes' *Géométrie* had appeared. Also, he was in possession, by 1628 or 1629, of his method for determining maxima and minima. Kepler had observed that the increment of a varying quantity became vanishingly small in the neighbourhood of its maximum or minimum values. Fermat translated this fact into a method for determining a maximum or minimum in the following way. If a function of x has a maximum value for the value x, then its value will be almost the same for $x-e$, if e is very small. If, therefore, the value $f(x)$ be equated to the value $f(x-e)$, we can make this equality correct by letting e assume the value zero. The roots of the equation so obtained for x will make $f(x)$ a maximum.

Fermat's first example is to divide B into two parts such that their product is a maximum. Let A and $B-A$ be the required parts. Now form

$$(A-E)(B-(A-E))$$

and equate the two products. From

$$A(B-A)=(A-E)(B-A+E)$$

we have

$$2AE-BE-E^2=0$$

or, dividing by E,

$$2A-B-E=0.$$

Now make E zero and we have $2A=B$, giving the required division.

It will be seen that Fermat's procedure is equivalent to forming

$$\left[\frac{f(x+h)-f(x)}{h}\right]_{h\to 0}=0,$$

i. e. to equating the first differential coefficient of a function to zero. The logic of his exposition leaves, of course, much to be desired, and his method does not distinguish between a maximum and a minimum value. Also, he did not know that the differential coefficient can vanish without corresponding to a maximum or minimum. He applied his method to determining the subtangent to the ellipse, cycloid, cissoid, conchoid, and quadratrix, by making the ordinates of the curve and a straight line equal for two infinitely near abscissae, x and $x-e$. Fermat also performed several integrations—on the principles, of course, of Cavalieri—evaluating the areas of parabolas and hyperbolas of different orders, and also obtaining the centre of mass of a paraboloid of revolution.

The work sketched above would be sufficient to rank Fermat amongst the greatest mathematicians of his time; but it is in the theory of numbers, a branch of mathematics he practically created, that he is unique. In this branch of mathematics he seems to have possessed a most extraordinary intuitive faculty. Many of his results were given without proofs, and some of the greatest succeeding analysts have been puzzled to supply them. We quote some theorems which were written as marginal notes in his copy of Bachet's *Diophantus.*

(1) A prime of the form $4n+1$ is only once the hypothenuse of a right triangle; its square is twice; its cube is three times, &c. Example: take

$$5=4\times 1+1. \quad 5^2=3^2+4^2; \quad 25^2=15^2+20^2=7^2+24^2;$$
$$125^2=75^2+100^2=35^2+120^2=44^2+117^2.$$

(2) A prime of the form $4n+1$ can be expressed once, and only once, as the sum of two squares. Euler obtained a proof of this.

(3) $x^4+y^4=z^2$ is impossible.

The theorem that an odd prime can be expressed as the difference of two square integers in one and only one way is given with a simple proof. Let n be the odd prime. Put $n=x^2-y^2$. Therefore $n=(x-y)(x+y)$. But n is prime. Its only factors are n and unity. Hence $x+y=n$ and $x-y=1$. Hence $x=\frac{1}{2}(n+1)$ and $y=\frac{1}{2}(n-1)$. What is sometimes called Fermat's last theorem states that no integral values of x, y, z can be found to satisfy the equation $x^n+y^n=z^n$ if n be an integer greater than 2. In the margin in which this theorem occurs it is accompanied by the note, 'I have found for this a truly wonderful proof, but the margin is too small to hold it.' In spite of repeated attempts by the keenest analysts, and the offer of several valuable money prizes by learned societies, the theorem still remains without proof. Euler has proved it to be true for $n=3$, and Fermat has elsewhere given an early proof for the case $n=4$. In 1823 Legendre gave a proof for $n=5$. In 1832 Dirichlet proved it for $n=14$, and Lamé and Lebesgue proved it for $n=7$ in 1840. In spite of the extraordinary difficulty of hitting upon a general proof, nobody doubts that the theorem is true, and it is generally supposed that Fermat really had found a proof. In a paper found amongst the manuscripts of Huygens in the library of Leyden in 1879, Fermat gives a general method by which he made many of his discoveries. He calls it *la descente infinie ou indéfinie.* In theorem (2) above, for instance, Fermat's proof proceeds by showing that if there is a prime $4n+1$ not possessing the property, then there is a smaller prime of the same form not possessing it. And, that being the case, there is yet a smaller prime of the same form not possessing it, and so on, until the number 5 is arrived at, which is the smallest prime of this form. But $5=2^2+1^2$ is the sum of two squares. Hence there is no prime of the form $4n+1$ not possessing the property.

PIERRE DE FERMAT

Another subject which engaged Fermat's attention was the almost entirely new question of the theory of probabilities. This branch of mathematics arose out of a correspondence between Pascal and Fermat respecting a problem proposed to the former by a gamester, the Chevalier de Méré. The problem may be stated thus: Two players of equal skill wish to give up a game before finishing it. Given the number of points necessary to finish the game and the score of each player, how are the stakes to be divided between them? Fermat discusses the case where

A wants two points to win and *B* three. Then four more trials will decide the game. Fermat takes two letters, *a* and *b*, and writes down all the combinations which can be formed by four of them. These are *aaaa*, *aaab*, *aaba*, *aabb*, *abaa...bbbb*. There are sixteen of them. Those cases where *a* appears two or more times are favourable to *A*. There are eleven of them. Those cases where *b* occurs three or more times are favourable to *B*. There are five of them. Hence the stakes should be divided in the proportion of 11 : 5. Pascal reached the same result by a different method. Another problem dealt with the probability of throwing a six with a die in eight throws. These would appear to be all the problems of this kind discussed by Fermat, but Pascal went on to consider other and more complicated cases.

BLAISE PASCAL

Blaise Pascal (1623–62) was a mathematical genius of a very high order, but he accomplished comparatively little in mathematics, since he devoted most of his manhood to religious meditation. At the age of twelve years he discovered, entirely without tuition, many elementary geometrical theorems, amongst them the theorem that the three interior angles of a triangle are together equal to two right angles. At the age of fourteen he was admitted to the weekly meetings of certain French mathematicians. Two years later he wrote an essay on the geometry of conics, which seems to have shown astounding ability. The greater part of this work is lost, but a report on its contents is contained in a letter by Leibniz. It appears that, at the age of sixteen, Pascal knew and appreciated Desargues' work, and was in a position to make fresh discoveries by the method. One of the results obtained by Pascal in this work is known as 'Pascal's theorem', and states that if a hexagon be inscribed in a conic, the points of intersection of the pairs of opposite sides will lie in a straight line.

In 1641, at the age of eighteen, Pascal invented the first arithmetical machine. About this time he turned his attention also to mechanics and physics. He repeated Torricelli's experiments, and showed that barometric readings really did depend on atmospheric pressure by obtaining, at the same moment, readings at different heights on the slope of the hill of Puy-de-Dôme.

This precocious and incessant activity seems to have had a bad effect on Pascal's health. Also, he had a strong vein of religious mysticism, and suddenly, in 1650, he abandoned all his mathematical and scientific researches in order to give himself up to religious contemplation. A tendency to religious and mystical speculation was not at all uncommon amongst mathematicians of this and preceding centuries. The forms in which it manifested itself were varied. Stifel, as we have seen, was obsessed by cabbalistic imbecilities; Newton devoted much thought and labour to the prophetic books of the Bible; while with Pascal this tendency received its expression in his magnificent *Pensées*, one of the treasures of French literature. In 1653, however, Pascal returned to the world and mathematics, but only for a short time. His mathematical power, which never failed him to the end of his life, was employed at this period in originating the calculus of probabilities, and in inventing the arithmetical triangle. He also did experimental work on the pressure exerted

JOHN WALLIS

by liquids and gases. But on November 23, 1654, Pascal received a very strong intimation that these activities were displeasing to God. On that date four runaway horses came within an ace of launching him over the parapet of the bridge at Neuilly. The miraculous breaking of the traces alone saved his life. Pascal pondered over this Divine hint, and, the better to guard against any more lapses, wrote out an account of it on a piece of parchment which he thenceforward wore next to his heart. The only mathematical work he produced after this was his essay on the cycloid, composed in 1658. The idea came into his mind when he was suffering from toothache, and directly the idea occurred to him his toothache stopped. Obedient, as he was, to every sign of the Divine will, Pascal immediately availed himself of the permission thus conveyed and worked incessantly at the problems presented by this curve for the next eight days, obtaining, in that time, a much more complete knowledge of its properties than had ever been obtained before, and solving problems that a subsequent challenge to the world showed that no other mathematician was able to solve. These problems concerned the positions of the centres of mass of the solids formed by rotating the curve about various axes, such as the axis of the curve, its base, and the tangent at its vertex. These problems were solved by Pascal by the method of indivisibles, and were equivalent to integrating $\sin \phi$, $\sin^2\phi$, $\phi \sin \phi$.

The mathematician who came nearest to solving the challenge questions issued by Pascal on the cycloid was *John Wallis* (1616–1703), who became Savilian Professor of Geometry at Oxford in 1649. He was a mathematician of remarkable ability and originality, particularly in analysis, and is one of those whose work most evidently prepared the way for Newton. He was the first to discuss conics as curves of the second degree, on the basis, of course, of Descartes' *Géométrie*, which few mathematicians had mastered, and in the following year, 1656, his analytical bent was still more clearly exhibited in his *Arithmetica infinitorum*, a book which was considered, for some years, the standard treatise on the subject. In this work Wallis reaches several remarkable results, often by deducing general propositions from a number of particular cases, a procedure not often logically rigorous but which, in his hands, usually led to correct results. It was known, for instance, that the curve $y=x^m$, where m is a positive integer, has an area which is to the area of the parallelogram having the same base and altitude as $1 : m+1$. Wallis stated that this formula held good even when m was fractional or negative. But in the case when m is negative and greater than unity Wallis was unable to interpret his results. If $m=-3$, for instance, Wallis could not understand what the ratio $1 : -2$ could mean. He concluded, however, that negative quantities were greater than infinity. He arrived at this by considering that $\frac{1}{a}$ gets larger and larger as a decreases, passing through infinity when $a=0$. Thus he considered that

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} < \frac{1}{0} < \frac{1}{-1}.$$

In notation he made a distinct advance. He introduces and explains negative and fractional, besides positive, indices. Thus he shows that

$$x^0 = 1\ ;\ \ x^{-1} = 1/x,\ x^{-2} = 1/x^2 \text{ etc.,}$$

that

$$x^{\frac{1}{2}} = \sqrt{x}\ ;\ \ x^{\frac{2}{3}} = \sqrt[3]{x^2}, \text{ etc.}$$

He also introduced the symbol ∞ for infinity.

We have already said that Wallis could effect summations equivalent to evaluating $\int_0^1 x^m\,dx$ for values integral and fractional of m. The manner in which he applied this knowledge to determine π is of great interest. His problem was to evaluate $\int_0^1 (1-x^2)^{\frac{1}{2}}\,dx$. He evaluated $\int_0^1 (1-x^2)^0\,dx$ and $\int_0^1 (1-x^2)^1\,dx$, as he could easily do by his formula. Now in $\int_0^1 (1-x^2)^{\frac{1}{2}}\,dx$ the exponent is the mean value between 0 and 1. The value of this integral might be taken, therefore, as the mean between the values of the integrals given above, i. e. 1 and $\frac{2}{3}$. But Wallis went on to form the series $\int_0^1 (1-x^2)^2\,dx$, $\int_0^1 (1-x^2)^3\,dx$, etc., obtaining the values $\frac{8}{15}$, $\frac{48}{105}$, etc. The question now became, what is that general law which, operating on the numbers 0, 1, 2, 3,... gives the numbers 1, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, ... and what would this law give if it operated on $\frac{1}{2}$? By an extremely difficult and complicated method Wallis arrived at the interesting expression for π,

$$\frac{4}{\pi} = \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdots\cdots}{2.4.4.6.6.8.....}$$

But Wallis did not succeed in his attempt at interpolation. This was left for Newton to do by his Binomial Theorem. Wallis showed his value for π to Lord Brouncker, first President of the Royal Society, who brought it into the form of a continued fraction

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \dots}}}}}$$

We may mention here that *Nicholas Mercator*, one of several mathematicians of this period who made minor advances, discovered the series

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + \dots$$

by division and, using Wallis's method of integration, obtained

$$\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \dots$$

These achievements excited much attention from the mathematicians of the time.

Many minor results, particularly in the theory of series and the problem of tangents, were obtained by various men at this time; but they are not of sufficient importance to warrant a detailed discussion, and practically all these particular cases were gathered up in the more general theorems soon to be enunciated by Newton. But before considering the masterful work of Newton we must give some account of the work of two men who were in a special sense his precursors.

Precursors of Newton

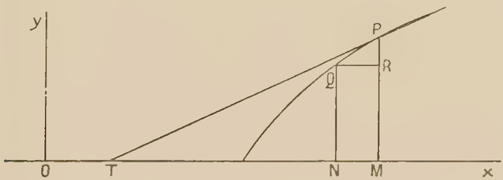
Christian Huygens (1629–95) was one of the greatest men that even this exceptionally rich century produced. In physics, mathematics, and astronomy his work is of first-rate importance. Our immediate concern is with his work in mathematics; but we may mention that he improved telescopes of which one consequence was his elucidation of the nature of Saturn's rings, and was the first to construct a watch whose motion was regulated by the balance spring. His most important work in physics was his exposition of the undulatory theory of light. By beautiful applications of geometry Huygens deduced, on his theory, the laws of reflection and refraction, and explained the phenomena of double refraction. He also did valuable experimental work.

But his most important work, from our present point of view, is his *Horologium oscillatorium*, published in 1673. In this work, amongst much other matter, the theory of evolutes is developed. He shows that the tangent of the evolute is normal to the involute and that, in the case of the cycloid, its evolute is an equal cycloid. In discussing this curve, a great favourite with the mathematicians of that time, Huygens made the beautiful discovery that it is tautochronous. This theorem occurs in the section devoted to the descent of heavy bodies, whether falling freely in a vacuum, or on inclined planes, or on certain smooth curves. He also investigates the compound pendulum, proving that the centres of oscillation and of suspension are interchangeable. In the last section of the book various theorems are proved relating to the centrifugal force on a body moving in a circle. He shows that this force, for a body moving with uniform speed, varies directly as the square of the speed and inversely as the radius of the circle.

CHRISTIAN HUYGENS

It is chiefly in the dynamical sections of this work that Huygens helped to prepare the way for Newton. Newton had, indeed, a quite special admiration for Huygens, and used to refer to him as the 'Summus Hugenius'. It is interesting, also, to note that both men were very fond of the methods of Greek geometry, with the result that much of their work wore a slightly archaic air even when it was first published. Huygens understood, however, the methods, so far as they were developed, of the infinitesimal calculus, and in one of his tracts improves upon Fermat's method of determining maxima and minima. He also made use, at times, of the co-ordinate methods of Descartes.

The other writer we must mention is *Isaac Barrow* (1630–77), a less gifted man than Huygens, but important for his probable influence on Newton. His *Mathematicae lectiones*, delivered from 1664 to 1666, are of little importance; but are worth mentioning for the doctrine they contain that a mathematical entity is to be regarded as built up out of similar entities—that a line is to be regarded, not as built up out of points, but out of small lines, and a surface not out of lines, but out of small surfaces. Barrow was Newton's teacher at Cambridge, and it was in 1669 that he resigned his chair in Newton's favour. In the same year appeared his most important work, the *Lectiones opticae et geometricae*. Barrow states in the preface that he is indebted to Newton for some of the matter in the book, and it is generally



supposed that Newton's contributions belong to the optical sections. By 1669 Newton began to communicate his method of fluxions to friends, but the chief point of interest in the book, the step forward it makes towards the creation of the differential calculus, was not influenced by Newton.

Barrow's method is distinguished by the innovation of importing two infinitesimal quantities. To obtain the tangent at the point P, for example, Barrow considers the small triangle PQR. Now when this triangle (which Barrow calls the differential triangle) becomes infinitely small the ratio PR/RQ is the same as the ratio PM/MT. Hence if the ratio PR/RQ can be determined when PR and RQ are infinitely small, the point T can be determined and hence PT, the tangent, can be drawn. Barrow's method is to take x, y as the co-ordinates of P and $x-e$, $y-a$ as the co-ordinates of Q. The co-ordinates for Q are related by the equation to the curve. Substituting these values, $x-e$, $y-a$, in the equation to the curve, and neglecting squares and higher powers of e and a, the ration of $a:e$ is obtained. Except that Barrow did not systematize his procedure and discover general rules it is evident from the above that he had practically created the differential calculus. The curves to which Barrow applied this method are

(a) $x^2(x^2+y^2)=r^2y^2$;

(b) $x^3+y^3=r^3$;

(c) $x^3+y^3=rxy$ (called by Barrow in a marginal note *La Galande*);

(d) $y=(r-x)\tan \pi x/2r$, the *quadratrix*;

(e) $y=r\tan \pi x/2r$.

To illustrate the method we will apply it to curve (b). Putting the co-ordinates of Q in the equation we have

$$(x-e)^3+(y-a)^3=r^3,$$

or

$$x^3-3ex^2+3e^2x-e^3+y^3-3ay^2+3a^2y-a^3=r^3.$$

Neglecting all powers of e and a higher than the first this becomes

$$x^3+y^3-3ex^2-3ay^2=r^3,$$

or, since

$$x^3+y^3=r^3,\ 3ex^2+3ay^2=0,$$

giving $a/e = -x^2/y^2$.

We have seen that, in the period covered by this and the preceding chapter, much has been accomplished. Co-ordinate geometry, projective geometry, the calculus of probabilities, the theory of numbers, the infinitesimal calculus, a sound system of dynamics, have all been more than begun. Of these branches of knowledge, undoubtedly the most important were the infinitesimal calculus and dynamics, the one as giving the most powerful instrument of research into almost all branches of mathematics, and the other as the necessary basis of the whole of applied mathematics, of astronomy and physics, in all their numerous branches. And in both these domains the accomplishments we have had to record hitherto were partial accomplishments. They were glimpses of the truth as it were. Mathematicians here and there were aware of one aspect or another of a great generalization which no one had fully grasped. It was a favourable moment for the advent of a really commanding genius, one who, as if standing at some extra elevation, should be able to combine in one clear vision all these partial aspects. And Providence, with a rare dramatic instinct, produced the genius in the person of Isaac Newton.

8

Newton

His Life

ISAAC NEWTON was born at Woolsthorpe, Lincolnshire, on Christmas Day, 1642, and died at Kensington, London, on March 20, 1727. He was the posthumous son of a yeoman farmer, and it was at first intended that Isaac also should devote his life to farming. It became evident, however, that this prospect did not attract the boy, and as his chief delight, both at home and at his school at Grantham, lay in devising all kinds of mechanical models, in making various sorts of experiments, and in solving problems, his mother sent him back to school and afterwards, in his eighteenth year, allowed him to enter Trinity College, Cambridge. Newton had not, at that time, read any mathematics. His attention was directed to the subject through the references to geometry and trigonometry he found in a book on astrology that he picked up at Stourbridge Fair. He began with Euclid, and was surprised to find that it contained nothing but what seemed perfectly obvious. He therefore ignored it and took up Descartes' *Géométrie*. This appears to have been difficult enough to interest him, and it cost him some little trouble to master it. He also read Oughtred's *Clavis*, and discovered that mathematics was so interesting a subject that he determined to make a serious study of it rather than of chemistry. He accordingly went on to read Kepler's *Optics*, the works of Vieta, and Wallis's *Arithmetica infinitorum*. By the early part of the year 1665, the year in which he took his B.A. degree, he had discovered the Binomial Theorem and invented his method of Fluxions, as he called what we now call the Differential Calculus. For part of the same and the succeeding year Newton lived at home to avoid the plague, and during this time he developed his calculus to a point which enabled him to find the tangent and radius of curvature at any point of a curve, and also applied it to the theory of equations. At the same time he took up various physical questions. He devised instruments for grinding lenses to other than spherical forms, made his first experiments on optics, and thought out the fundamental principles of his theory of gravitation.

Newton returned to Cambridge in 1667 and was elected a Fellow of his college. For the next two years his time was chiefly occupied with his optical researches. In October 1669 Barrow resigned his professorship in favour of Newton. Newton's first lectures were on optics and dealt with the results of his own researches. These were embodied in a paper communicated to the Royal Society in 1672, and roused great interest and much discussion. That he should be involved in discussions, about anything whatever, was always abhorrent to Newton, and in the present case had the unfortunate effect of making him resolve never again to publish anything. Thus, in a letter of 1675 he says, 'I was so persecuted with discussions arising out of my theory of light, that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow.' In a letter of a year later the matter is still rankling. He writes, 'I see I have made myself a slave to philosophy; but, if I get rid of Mr. Lucas's business, I will resolutely bid adieu to it eternally, excepting what I do for my private satisfaction, or leave to come out after me; for I see a man must either resolve to put out nothing new, or to become a slave to defend it.' It certainly seems to be true that one of Newton's strongest desires was to be left alone. He never exploited his unparalleled genius in order to win prestige, either social or scientific. The whole of the unfortunate controversy with Leibniz respecting the invention of the differential calculus could never have arisen if Newton had published his results at the time he obtained them. As for social prestige, it appears that, even as a young man, he showed very little interest in anything but his scientific studies, and was completely indifferent to whatever notoriety his gifts might have brought him. Thus when, in 1669, he had solved some problems, at Collins's request, which had proved insoluble to other mathematicians, he insisted that the results, if published, should be published without his name appearing, 'for I see not what there is desirable in public esteem, were I able to acquire and maintain it: it would perhaps increase my acquaintance, the thing which I chiefly study to decline.' This characteristic of Newton's has an important bearing on the history of mathematics, for it was owing to this dislike of publicity that the controversy with Leibniz arose, and it was owing to that controversy that English mathematicians cut themselves off from the Continental school, and it was owing to this separation that the progress of mathematics in England was appreciably retarded for the best part of a hundred years.

ISAAC NEWTON

Newton continued to work at optics, and by the end of 1675 he had worked out his emission or corpuscular theory of light. His results were communicated to the Royal Society in December of that year. Newton's theory is very ingenious, and he succeeds in explaining a large number of optical phenomena. His reputation led to his theory being generally accepted, and it was not until many years later that Huygens's undulatory theory was shown to be a better instrument of research. From 1673 to 1683 Newton lectured on algebra and the theory of equations. We shall refer to some of his results in this field later. In 1679 Hooke, at the request of the Royal Society, wrote asking Newton for further contributions, and received a reply saying that Newton had abandoned the study of philosophy. As a matter of fact Newton was devoting much of his time to theology and other subjects—studies which interested him all his life. Hooke had mentioned in his letter Picard's new geodetical researches, and had given Picard's value for the radius of the earth. With these new data Newton took up his old calculations on gravitation and verified, from the motion of the moon, his inverse square law. He also proved that if a body describes an ellipse under the influence of a force directed to a focus, that force must vary as the inverse square of the distance. Also, a body projected under the influence of such a force would describe an ellipse (or, more generally, a conic). Having obtained these extremely important results, Newton simply kept them to himself.

Five years later, in August 1684, Halley came to Cambridge to consult Newton about a problem. It appeared that some of the keenest intellects of the time, Huygens, Hooke, Halley, Wren, all had a shrewd suspicion that the orbits of the planets pointed to the existence of a force directed towards the sun, and varying inversely as the square of the distance. But it remained a suspicion, for they found themselves unable to demonstrate that such a law of force would give an elliptical orbit. It was on this problem that Halley came to consult Newton. Newton immediately replied, of course, that such a law of force would give an elliptical orbit, but was unable to find his five years' old demonstration. He promised, however, if he were unable to find it, to work it out afresh. In November Halley received the demonstration. But the conversation with Halley had revived Newton's interest in the matter, and he had worked out, before the autumn, many of the fundamental propositions of the first book of his *Principia*. In his communication to Halley Newton incorporated two or three of these results, with the result that Halley again went to Cambridge. He there saw the manuscript of Newton's recent work and realized its immense importance. Halley appears to have had considerable influence with Newton; for he not only got him to promise to send these results to the Royal Society, but probably prevailed upon him really to devote himself to working out the whole theory of gravitation. Halley, on his return to London, informed the Royal Society of what he had seen and of the promise he had received from Newton. Halley and Mr. Paget, a Fellow of Newton's own college, were accordingly instructed by the Royal Society 'to put Mr. Newton in mind of his promise', with the result that by the middle of February Newton sent a paper containing his 'notions about motion' to the Royal Society. About this time, probably a little before, Newton had worked out the extremely important and beautiful theorem that a spherical body whose density at any point depends only on its distance from the centre, attracts an external point as if its whole mass were concentrated at its centre. The discovery of this theorem meant that the inverse square law could be rigorously applied to the members of the solar system, treating the sun and planets as heavy particles, since the slight departures from exactitude, due to the fact that the heavenly bodies were not perfect spheres, could obviously be neglected as being much below the limits of observation.

Newton was now, therefore, in possession of all the fundamental theorems he required, and he addressed himself in earnest to the task of working out the implications of his great theory. By an almost incredible intellectual effort he had composed the first book of the *Principia* by the summer of that year, 1685. This feat has been called the greatest single illustration of Newton's powers. By the summer of the next year the second book was completed and the third begun. But the communication of the first book had aroused a discussion. Hooke, a very able but vain and jealous man, claimed to have preceded Newton in discovering the inverse square law. As we have seen, he had surmised it, but was incapable of proving it. Newton deals in some detail with Hooke's claim in a letter to Halley, mentions that he had designed the whole work to consist of three books, and then makes the alarming statement: 'The third I now design to suppress. Philosophy is such an impertinently litigious lady, that a man has as good be engaged in lawsuits, as have to do with her. I found it so formerly, and now I am no sooner come near her again, but she gives me warning.' Halley, who had taken upon himself the cost of printing and publishing the work, who was more responsible for it than any man except the author, and who probably knew better how to handle this difficult genius than did anybody else, wrote an imploring letter begging Newton not to withhold the third book, and informing him that the Royal Society had agreed that Hooke's claim could not be allowed. Newton, whose motive on this and other occasions does not appear to have been vanity, but rather a cold and scrupulous justice, thereupon agreed to publish the third book and added a note, 'The inverse law of gravity holds in all the celestial motions, as was discovered also independently by my countrymen Wren, Hooke, and Halley.' This and other difficulties being smoothed over, the complete treatise, entitled *Philosophiae naturalis principia mathematica*, was published about midsummer of 1687. It immediately made a great impression throughout Europe, and the whole edition was quickly sold.

In 1689 Newton was elected member of Parliament for the university, but the Parliament only lasted thirteen months, and in 1690 Newton returned to Cambridge and renewed his mathematical studies. Towards the close of 1692 Newton was afflicted with a curious illness which lasted about two years. Its precise nature is unknown, but there is no doubt that it led to some form of mental derangement. After this period he was practically lost to science. He did not lose his power, for very late in life he showed the old mastery in solving problems submitted to him, but he seems to have done very little work on his own initiative. And in any case his appointment, first as warden, and afterwards, in 1699, as Master of the Mint, was practically equivalent to his abandonment of scientific studies. In 1703 he was elected President of the Royal Society, and was annually re-elected till his death. In 1705 he was knighted. From now on he seems to have occupied his leisure chiefly with theology, chemistry, and alchemy. He wrote a good deal on these subjects, and was also an attentive student of mysticism. Copious extracts from the writings of Jacob Boehme, in Newton's writing, were found amongst his manuscripts. At one time he spent several months searching for the philosopher's stone. Besides these activities he was much worried, for some years after 1707, by his controversy with Leibniz. He died after a lingering and painful illness, and eight days later was buried in Westminster Abbey.

His Works

Newton's chief published works, all except the *Principia*, appeared years after the results embodied in them had been discovered; even so, it took mathematicians quite half a century to assimilate them. It will be useful, in discussing these writings, to give a list of their dates of publication. *Principia*, 1687;

Optics (with appendices on *cubic curves*, the *quadrature and rectification of curves by the use of infinite series*, and the *method of fluxions*), 1704; *Universal Arithmetic*, 1707; *Analysis per Series, Fluxiones, etc.*, and the *Methodus Differentialis*, 1711; *Lectiones Opticae*, 1729; the *Method of Fluxions, etc.*, translated by J. Colson, 1736.

Besides these works we must mention two letters to Oldenburg, written in 1676, in which Newton describes some of his mathematical methods, and gives some information about the way he discovered them. We have already mentioned that Wallis, from the series

$$\int(1-x^2)^0 dx = x\,;\ \int(1-x^2)^1 dx = x-\tfrac{1}{3}x^3,$$

$$\int(1-x^2)^2 dx = x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5,\ \int(1-x^2)^3 dx = x-\tfrac{3}{3}x^3+\tfrac{3}{5}x^5-\tfrac{1}{7}x^7,$$

and so on, had endeavoured, by interpolation, to obtain the expression for $\int(1-x^2)^{\frac{1}{2}} dx$, but had been unsuccessful. It seemed to Wallis that the desired expression must have more than one term and less than two. Newton, as we have seen, had read Wallis's work, and it occurred to him that the interpolation could be arrived at in the following way. In each expression the first term is x; x increases in odd powers; the signs are alternately + and −; the second terms $\frac{0}{3}x^3, \frac{1}{3}x^3, \frac{2}{3}x^3, \frac{3}{3}x^3$, are in arithmetical progression; the denominators 1, 3, 5, 7, &c., are in arithmetical progression. He accordingly wrote for the first two terms of the interpolated series $x - \dfrac{\frac{1}{2}x^3}{3}$. He also saw that the coefficients in the numerators of each expression are the digits of successive powers of the number 11. Thus 11^0 gives 1, the only numerator in the first expression; 11^1 gives 1, 1, the numerators in the second expression; 11^2 gives 1, 2, 1, the numerators in the third expression; 11^3 gives 1, 3, 3, 1, and so on. But this will not give a result when the power is $\frac{1}{2}$. Newton then saw that, calling the second digit m, the remaining digits could be found by continued multiplication of the terms of the series $\frac{m}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}$, &c. Thus, in the expression $x - \frac{3}{3}x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7$, m, the numerator of the second term, is 3. Accordingly the numerator of the third term is $\frac{3}{1} \cdot \frac{3-1}{2} = 3$, and of the fourth term is $\frac{3}{1} \cdot \frac{3-1}{2} \cdot \frac{3-2}{3} = 1$. For the interpolated series $m = \frac{1}{2}$. Hence the coefficient in the numerator of the third term is $\frac{1}{2} \cdot \frac{\frac{1}{2}-1}{2} = -\frac{1}{8}$; for the fourth term it is $-\frac{1}{8} \cdot \frac{\frac{1}{2}-2}{3} = \frac{1}{16}$; for the fifth term it is $\frac{1}{16} \frac{\frac{1}{2}-3}{4} = -\frac{5}{128}$, and so on. The interpolated series therefore becomes

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} - \dots$$

Newton then applied this method to obtain

$$(1 - x^2)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \dots$$

He verified this result by extracting the square root of $(1 - x^2)$, and also by multiplying the above series by itself. His generalization of this result, the Binomial Theorem, he enunciated in the form

$$(P + PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \dots$$

where A means the first term, viz., $P^{\frac{m}{n}}$, B means the second term, viz., $\frac{m}{n}AQ$, C means the third term, and so on. This discovery belonged, as we have said, to the earliest part of Newton's career. He used it to integrate, by Wallis's method of quadratures, functions of x which could be expressed in power series of x. The other and more important discovery he made about the same time was his method of Fluxions. The essentials of this method he communicated to Barrow in 1669. In 1671 he wrote his *Method of Fluxions*, but for some reason, possibly his dislike of publicity, did not publish it.

It is difficult to say exactly how far Newton developed his calculus, or at what precise dates the different developments occurred. In the following account we shall rely chiefly on Colson's translation of Newton's Latin manuscript. Newton regarded geometrical magnitudes as being generated by motion. In the *Quadrature of Curves* he states that: 'I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described and thereby generated, not by the apposition of parts, but by the continued motion of points; superfices by the motion of lines; solids by the motion of superfices; angles by the rotation of their sides; portions of time by continual flux: and so on in other quantities.' This is the basic idea that Newton seems to have had from the beginning. But the notion of time was not as fundamental in his conception as from the above passage it would appear to be. In the *Method of Fluxions* he says: 'But whereas we need not consider the time here, any farther than it is expounded and measured by an equable local motion; and besides, whereas only quantities of the same kind can be compared together, and also their velocities of increase and decrease; therefore, in what follows I shall have no regard to time formally considered, but I shall suppose some one of the quantities proposed, being of the same kind, to be increased by an equable fluxion, to which the rest may be referred, as it were to time; and, therefore, by way of analogy, it may not improperly receive the name of time.'

Growing quantities were called by Newton *fluents*, and their rates of growth were called the *fluxions* of the fluents. In his own words: 'Now those quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *fluents*, or *flowing quantities*, and shall represent them by the final letters of the alphabet, v, x, y, and z;—and the velocities by which every fluent is increased by its generating motion (which I may call *fluxions*, or simply velocities, or celerities), I shall represent by the same letters pointed, thus, $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$. That is, for the celerity of the quantity v I shall put $\dot{v}$, and so for the celerities of the other quantities x, y, and z, I shall put $\dot{x}$, $\dot{y}$, and $\dot{z}$ respectively.' Newton now introduces another conception, what he calls the *moments* of flowing quantities. The moment of any fluent is defined to be its fluxion multiplied by an infinitely small quantity. Newton writes o for this infinitely small quantity. Thus the moment of x is $\dot{x}o$. 'The moments of flowing quantities (that is, their indefinitely small parts, by the accession of which, in infinitely small portions of time, they are continually increased) are as the velocities of their flowing or increasing. Wherefore, if the moment of any one (as x) be represented by the product of its celerity $\dot{x}$ into an infinitely small quantity o (i.e. by $\dot{x}o$), the moments of the others, v, y, z, will be represented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$ are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

'Now, since the moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing quantities x and y, by which those quantities are increased through the several indefinitely little intervals of time, it follows that those quantities, x and y, after any indefinitely small interval of time, become $x+\dot{x}o$ and $y+\dot{y}o$, and therefore the equation, which at all times indifferently expresses the relation of the flowing quantities, will as well express the relation between $x+\dot{x}o$ and $y+\dot{y}o$, as between x and y; so so that $x+\dot{x}o$ and $y+\dot{y}o$ may be substituted in the same equation for those quantities, instead of x and y. Thus let any equation $x^3-ax^2+axy-y^3=0$ be given, and substitute $x+\dot{x}o$ for x, and $y+\dot{y}o$ for y, and there will arise

$$\left.\begin{array}{l} x^3+3x^2\dot{x}o+3x\dot{x}o\dot{x}o+\dot{x}^3o^3 \\ -ax^2-2ax\dot{x}o-a\dot{x}o\dot{x}o \\ +axy+ay\dot{x}o+a\dot{x}o\dot{y}o \\ +ax\dot{y}o \\ -y^3-3y^2\dot{y}o-3y\dot{y}o\dot{y}o-\dot{y}^3o^3 \end{array}\right\} = 0.$$

Now, by supposition, $x^3-ax^2+axy-y^3=0$, which therefore, being expunged and the remaining terms being divided by o, there will remain

$$3x^2\dot{x} - 2a\dot{x}x + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} + 3x\dot{x}\dot{x}o - a\dot{x}\dot{x}o$$
$$+ a\dot{x}\dot{y}o - 3y\dot{y}\dot{y}o + \dot{x}^3oo - \dot{y}^3oo = 0$$

But whereas zero is supposed to be infinitely little, that it may represent the moments of quantities, the terms that are multiplied by it will be nothing in respect of the rest; therefore I reject them, and there remains

$$3x^2\dot{x} - 2a\dot{x}x + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0.\text{'}$$

Newton proposed to himself two classes of problems. The first was, 'the relation of the fluents being given, to find the relation of their fluxions,' i.e. to differentiate. The second was, 'an equation being proposed exhibiting the relation of the fluxions of quantities, to find the relations of those quantities, or fluents, to one another,' i.e. to solve differential equations. Of the second class of problems Newton considered three types, (1) when the equation contains two fluxions of quantities, and but one of the fluents, (2) when the equation contains both fluents and both fluxions, (3) when the equation contains the fluents and the fluxions of three or more quantities. The first type may be expressed as $\frac{dy}{dx} = f(x)$. The second type includes any differential equation of the first order. Newton's solutions were obtained in the form of infinite series. The third type comprises partial differential equations. Newton obtained a particular integral of $2\dot{x} - \dot{z} + x\dot{y} = 0$. In this treatise Newton makes many geometrical applications of his calculus. He determines maxima and minima, tangents to curves, curvature of curves, rate of increase of the radius of curvature, quadrature, and rectification of curves.

So far Newton uses infinitely small quantities. He later tried to establish the calculus without them, but his language is obscure, and it is not clear whether or not he did reach the modern notion of limits. In the applications of his calculus Newton showed very great ability, and some of his integrations, in particular, show extraordinary mathematical acumen.

The *Arithmetica Universalis*, which contains the substance of the lectures delivered from 1673 to 1683, is chiefly concerned with algebra and the theory of equations. It contains a variety of results of great importance, such as that imaginary roots must occur in pairs, rules for findng the superior limit to the positive roots of a numerical equation, the theorem for finding the sum of the nth powers of the roots of an equation, and his mysterious rule for finding the number of imaginary roots of an equation. In some cases, where Newton merely enunciates his theorems, the proofs were not discovered for several years. The rule for finding the number of imaginary roots, for instance, was not thoroughly understood till 1865, when Sylvester published a paper on the subject.

A work which shows a different aspect of his many-sided mathematical genius is his *Enumeratio linearum tertii ordinis*, which investigates the properties of cubic curves, apparently as an exercise in analytical geometry. He classifies all the cubics, and enumerates seventy-two out of the possible seventy-eight forms that a cubic may take. Most of his theorems were given without proof, and the most baffling of them is his assertion that just as the shadow of a circle gives rise to all the conics, so the shadows of the curves $y^2 = ax^3 + bx^2 + cx + d$ give rise to all the cubics. This statement was not proved for several years, when Nicole, Clairaut, and Murdoch gave demonstrations. Newton's greatest work is undoubtedly his *Principia*. In this work a really complete system of dynamics was expounded for the first time and, in conjunction with the theory of universal gravitation, a complete mathematical formulation was given of the chief phenomena of motion, both terrestrial and celestial. It would be impossible, within our space, to describe the investigations contained in this treatise. No other work in the history of science has had so great an influence or has been so generally admired. It is not too much to say that every great mathematician since has regarded it as the supreme manifestation of mathematical genius. Lagrange stated that he felt dazed at such an illustration of what man's intellect might be capable of—a statement the more impressive in virtue of the ordinary mathematician's dazed feeling when faced by Lagrange's own work. But we must call attention to one very profound and far-reaching consequence of Newton's work which has only been realized within our own day. The whole of the Newtonian dynamics presupposes a privileged frame of reference with respect to which the fundamental laws of motion hold good. From this assumption springs the concept of force, and, in particular, of the force of gravitation. The assumptions made by Newton seem so natural that hardly anybody, until the most recent times, seems to have realized that they were assumptions. The whole of physics and astronomy, as it has been developed hitherto, rests on the Newtonian assumptions. In this respect Newton's formulation of a dynamical scheme had an influence more profound and more unquestioned than any other of his achievements. Just as the creation of the non-Euclidean geometries showed us the real status of Euclid's axioms, so the creation of the theory of relativity has shown us the real status of Newton's laws of motion.

In considering the whole of Newton's work we see that the immense reputation he has always enjoyed is fully deserved. He was a superb analyst and, although he created nothing new in pure geometry, nobody of his time or since has shown his power in using geometrical methods. On both counts he is one of the supreme mathematicians of the world. In addition, his insight into physical problems and his ability to give them mathematical expression have never been excelled. And he was also a great experimentalist. It is the union of these qualities which makes Newton unexampled in the history of science.

9

The Invention of the Calculus

THE notation of the differential and integral calculus, in the form we now have it, is due to *Gottfried Wilhelm Leibniz* (1646–1716) one of the most distinguished philosophers and mathematicians Germany has ever produced. His chief title to consideration, as a mathematician, lies in his claim to be an independent inventor of the calculus. This claim is now admitted, by the majority of authorities, but during the eighteenth century it was generally considered that Leibniz was merely a plagiarist of Newton. It was the long and occasionally fierce controversy on this point that led to the unfortunate separation of the English from the Continental school that we have already mentioned. There is no question that Newton was the first inventor of the calculus, nor that Leibniz originated the notation we now use. The only question is whether or not Leibniz borrowed the fundamental ideas of the calculus from Newton. The matter is involved, and its proper discussion would be lengthy, but, owing to its historical importance, we will mention the main points.

Leibniz began the serious study of mathematics rather late. It was in 1672 that he met Huygens in Paris, and was incited by him to study geometry. Leibniz had previously written on mathematics, but nothing of importance. By 1674, according to his own statement, he invented the differential and integral calculus. The earliest traces of it to be found in his note-books occur in 1675. He had not developed it into a system until 1677. It was published in 1684. In 1676 Leibniz was in correspondence with Oldenburg, secretary of the Royal Society, on the subject of infinite series. Oldenburg mentioned that Newton had obtained some important results, whereupon Leibniz asked for further information. The result was two letters from Newton, addressed to Oldenburg, and written 13 June and 24 October 1676. The letters contain a wealth of theorems, but the method of fluxions is only referred to in two anagrams. The first is

6accdae13eff7i3l9n4o4qrr4s8t12vx

which, being read, is 'Data aequatione quotcunque fluentes quantitates, involvente, fluxiones invenire, et vice versa.' The second anagram, another collection of letters, means 'Una methodus consistit in extractione fluentis quantitatis ex aequatione simul involvente fluxionem eius: altera tantum in assumptione seriei pro quantitate qualibet incognita ex qua caetera commode derivari possunt, et in collatione terminorum homologorum aequationis resultantis, ad eruendos terminos assumptae seriei'. This last anagram refers to the 'inverse problem of tangents'.

It is obvious that Leibniz could have learned nothing from these anagrams. In his reply, dated 21 June 1677, Leibniz explains his calculus, introduces the notation *dx* and *dy*, and illustrates its use. There was no hint at this time that Leibniz's calculus was not an independent invention. In the first edition of the *Principia* Newton says: 'In letters which went between me and that most excellent geometer, G. G. Leibniz, ten years ago, when I signified that I was in the knowledge of a method of determining maxima and minima, of drawing tangents, and the like, and when I concealed it in transposed letters involving this sentence [quoted above], that most distinguished man wrote back that he had also fallen upon a method of the same kind, and communicated his method, which hardly differed from mine, except in his forms of words and symbols.'

In 1699 *Fatio de Duillier*, a rather unimportant Swiss mathematician, strongly hinted, in the course of a paper communicated to the Royal Society, that Leibniz had stolen the idea of the calculus from Newton. Leibniz protested energetically against this slander, and was apparently pacified when informed that the Royal Society had not noticed this passage in Fatio's paper and did not approve of it. In 1704 Newton's *Optics* was published, with the *Quadratura Curvarum* as an appendix. This appendix was reviewed in Leibniz's periodical, the *Acta Eruditorum*, in 1705. Leibniz wrote the review, which was published anonymously, and in the course of it, after referring to himself as the inventor of the differential and integral calculus, says that, instead of Leibniz's differentials Newton uses, and always has used, fluxions, and that he has made elegant applications of them, just as Fabri applied the method of Cavalieri. The intention of this passage was obvious. Fabri was an entirely insignificant person who applied the methods of Cavalieri, and the parallel only makes sense if Leibniz takes the place of Cavalieri and Newton of Fabri. The assertion is that Leibniz invented the calculus and that Newton adopted it, called it by another name, and made 'elegant' applications of it. Leibniz always denied this meaning of the passage, but it is obvious, considering that Leibniz was a very clever man and a practised writer, that it can bear no other.

The passage was understood in its obvious sense by *John Keill*, a Scottish mathematician, and professor of physics at Oxford, and roused him to make the obvious retort. In a paper communicated to the Royal Society in 1708 he remarks, apropos of nothing in particular, that Newton invented the method of Fluxions which was afterwards, with a changed name and notation, published by Leibniz. Leibniz, of course, strongly protested against this, but the Royal Society, instead of disavowing Keill, asked him to justify his statement if possible. Accordingly Keill wrote a long letter making his slander rather clearer and more emphatic than before. As a result Leibniz appealed to the Royal Society to judge of this matter. A committee was appointed and the whole question thoroughly investigated. The findings of the committee were published in a document called the *Commercium epistolicum*, issued in 1712. Detailed references, with dates and names, are given, the bulk of the material examined being letters to and from Newton, Leibniz, Wallis, Collins, &c. The formal judgement reached by the committee was to the effect that Newton was the first inventor of the calculus. This left the real question in dispute unresolved. The whole spirit of the document, however, was hostile to Leibniz. This unsatisfactory attitude on the part of the Royal Society merely made the controversy more bitter. It went on for years, but without much fresh light being thrown on the subject. The whole question, as it stands now, may be summarized by saying that, where Leibniz's personal testimony is concerned, the highest value is not attributed to it. It has been proved that he was not above falsifying dates and making alterations in important documents. Also, it has been shown to be probable that he did have access to some of Newton's manuscript writings at a date preceding his own development of the calculus. Authorities are not agreed, however, as to the possibility of even so gifted a man as Leibniz penetrating to the idea of the calculus from the hints given by Newton. And great weight must be attached to the fact that Leibniz's note-books reveal a gradual and natural development of his ideas on the calculus. It is not, *a priori*, at all incredible that Leibniz should be an independent inventor. As we have seen, several earlier writers had come near to its main idea. It is not at all unlikely that there should have existed two men with sufficient ability to make the step forward that was required.

As a mathematician Leibniz's fame rests chiefly on his papers on the calculus. He was a mathematician of remarkable analytical skill, and abounded in ideas. Thus he introduced the method of indeterminate coefficients, and was the first to use determinants. In two papers he laid the foundations of the theory of envelopes—in one of these there occur for the first time the terms *co-ordinates* and *axes of co-ordinates*. He also wrote on osculating curves, and on dynamics. But he was capable of grave blunders, and although he solved some important problems in his dynamical papers, yet his writings show that, even twenty years after the publication of Newton's *Principia*, his fundamental ideas on the subject were still confused. As compared with Newton Leibniz is, on the whole, inferior as an analyst, while in geometrical power and in his insight into the mathematical laws of natural phenomena he is much below Newton. Nevertheless Leibniz had a superior instinct for mathematical form, and it is for that reason that his notation for the calculus is so much better than Newton's. The same instinct led him to seek solutions in finite form instead of contenting himself, as did Newton, with solutions in infinite series. On both these grounds the modern development of the calculus owes much more to Leibniz than to Newton.

The fundamental principles of the calculus, as expounded by both Newton and Leibniz, remained obscure. Leibniz used infinitesimals, and spoke of quantities being small enough to neglect. Newton at one time also used this language, but later states that no quantities, however small, are to be neglected. The alternative conception that he gave is obscure ; it would appear that he was working towards the modern conception of a limit, but that he never fully attained it. This fundamental obscurity in the new calculus was the subject of many attacks, which the disciples, both of Newton and Leibniz, exerted themselves to meet. Progress was thereby made, although the whole subject was not placed upon impeccable logical foundations till many years later. The history of this question is, indeed, a very interesting illustration of the way in which, even in mathematics, a correct intuition may commend itself to others and lead to a great wealth of new knowledge, long before a logical proof of its correctness can be supplied, and before the intuition itself can be correctly formulated.

With the invention of the calculus a new era in mathematics was begun. No other single mathematical invention has proved so fruitful. The period immediately following the death of Leibniz is remarkable chiefly for the developments and applications of the calculus. There were some amongst Leibniz's contemporaries, in particular James and John Bernoulli, who had already begun to show how powerful and flexible an instrument the new calculus was—a movement which went on with steadily increasing momentum up to the time of Lagrange and Laplace. And from Lagrange and Laplace we pass, with no breach of continuity, to the mathematics of our own time. On the broadest historical survey, then, we may choose the invention of the calculus as inaugurating the modern period and terminating the old.

Supplementary Chapter

Summary of the progress from Newton to the Introduction of Rigorous Methods

THE logic of the differential and integral calculus, as it left the hands of Newton and Leibniz, was not satisfactory. The logic of many other branches of mathematics, also, was not satisfactory. Newton and Leibniz were almost alone, for instance, in realizing that an infinite series should be convergent. But they had no satisfactory criteria for convergence, and most mathematicians, for many years after the death of Leibniz, used infinite series in a manner which seems to us surprisingly rash. The notions of continuity and discontinuity were also in a nebulous state, and even negative and imaginary quantities continued to present conceptual difficulties. From the time of Leibniz to that of Gauss mathematicians were too busy in exploiting the wonderful powers of the new calculus to pay much attention to the logical foundations of their science. There were exceptions. Here and there one finds a man who was uneasy in the presence of a merely formal development, but the characteristic features of this period are imaginative vigour and blind confidence.

The English school, as we have said, pursued a line of its own. It is probably unfortunate that Newton chose to use classical geometrical methods in his *Principia* instead of the calculus, for his success helped to persuade some of his followers that the calculus was not really necessary. This hindered its development in England, as did also the fact that the Newtonian notation was not really convenient, while the unfortunate controversy with Leibniz made it a point of honour with English mathematicians to learn nothing from the Continent. Amongst the English contemporaries and immediate successors of Newton were some very able men, of whom we may particularly mention *Brook Taylor* (1685–1731) and *Colin Maclaurin* (1698–1746).

Taylor is chiefly remembered for his well-known expansion of a function of a single variable in powers of the variable,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots$$

but in his own proof of it he pays no attention to the convergency of the series, and makes a number of unjustified assumptions. A rigorous proof was given by Cauchy. Taylor also gave formulae for the change of the independent variable in differentiation. Thus, expressed in modern notation, he obtained the formula

$$\frac{d^2x}{dz^2}=-\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^3}$$

and the corresponding formulae for

$$\frac{d^3x}{dz^3},\ \frac{d^4x}{dz^4}.$$

Maclaurin obtained the expansion, which can be immediately derived from Taylor's,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots$$

but he, also, did not investigate the convergency of the series. Maclaurin was chiefly remarkable for his immense ability in using the methods of classical geometry, which he applied to numerous problems in geometry, statics, the theory of attractions, and astronomy. This part of his work caused Clairaut to abandon analysis, and to attack the problem of the figure of the earth by pure geometry. It also aroused great admiration in Lagrange, who compared it with the best achievements of Archimedes. Its influence on his countrymen was to give them additional encouragement to neglect the development of the calculus. Amongst other mathematicians of the English school we may mention *Roger Cotes*, who died young, and of whom Newton said 'If Cotes had lived, we might have known something', *Abraham Demoivre*, famous for his creation of the trigonometry of imaginary quantities, and *Thomas Simpson*, the last notable English mathematician of the eighteenth century. He died in 1761, and no other English mathematician who requires mention appeared for the better part of a century.

On the Continent, however, the period from Leibniz to Gauss was one of the richest in the history of mathematics. Contemporary with Leibniz, and immediately succeeding him, were the members of the very remarkable *Bernoulli* family. We need here mention only the two brothers *James* (1654–1705) and *John* (1667–1748), and *Daniel* (1700–82), the most gifted of the three sons of John. It was chiefly owing to the influence of James and John that the calculus, in Leibniz's notation, became known all over the Continent. They used it to solve many important problems, and developed it systematically. Besides this, they created various branches of analysis. Daniel's chief work is his *Hydrodynamica*, published in 1738; all the results are deduced from the principle of the conservation of energy. The so-called 'Bernoulli numbers' was discovered by James Bernoulli. They occur in his *Ars Conjectandi*, a treatise on the calculus of probabilities.

The first treatise on the new calculus was the *Analyse des infiniment petits* of *Guillaume François Antoine l'Hospital, Marquis de St.-Mesmo* (1661–1704) an early pupil of John Bernoulli. This treatise did much to extend the popularity of the new calculus, particularly in France. A really satisfactory treatise did not appear, however, till 1742, when Maclaurin published his *Treatise of Fluxions*, which was designed to avoid the acute objections raised against the principles of the calculus by Bishop Berkeley.

Two other French mathematicians of this period must be named, *Alexis Claude Clairaut* (1713–65) and *Jean le Rond D'Alembert* (1717–83). Clairaut was a striking instance of the infant prodigy, since he read l'Hospital's work on the calculus and also his conic sections at the age of ten. His most remarkable work is his *Théorie de la figure de la Terre*. This was published in 1743. In 1752 he published his *Théorie de la Lune*, in which the motion of the lunar apsides is explained. The general theory of dynamics was greatly advanced by the *Traité de dynamique* of D'Alembert, published in 1743, and containing his celebrated principle that the impressed forces are equivalent to the effective forces. In 1744 he applied this principle to the equilibrium and motion of fluids, and two years later to the general causes of winds. In 1747 he discussed the problem of a vibrating string. Each of these researches had led him to a differential equation of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{dx^2}$$

and he now gave as its solution $u = \phi(x+t) + \psi(x-t)$, when ϕ and ψ are arbitrary functions. D'Alembert also made valuable contributions to astronomy, and in particular solved the problem of the precession of the equinoxes.

A mathematician with a singularly flamboyant imagination was *Johann Heinrich Lambert* (1728–77), whose work was rather overshadowed by that of his great contemporaries. It may be mentioned, however, that he proved that π is irrational, and introduced the hyperbolic sine and cosine, which he denoted by $\sinh x$ and $\cosh x$, into analysis.

There are many other mathematicians who would require mention in any less summary account of this period, but we must devote the bulk of our remaining space to an account of the work of the three great masters, Euler, Lagrange, Laplace. *Leonhard Euler* was born at Bâle in 1707, and died at Petrograd in 1783. He was a mathematician of prodigious inventiveness and industry, and a large part of his work consisted in completing and arranging practically every branch of mathematics then known. In addition, he created many new notions and procedures. He wrote extensively on the calculus, differential geometry, and infinite series. He invented the Beta and Gamma functions, investigated elliptic integrals, discovered the elements of the calculus of variations, and obtained new results in the theory of numbers. But Euler, like the other mathematicians of this century, was not rigorous. He occasionally warns his readers that infinite series should be convergent, but his own way of handling infinite series is often very loose. Thus he finds that

$$\ldots \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \ldots = 0$$

by adding the two series

$$n + n^2 + \ldots = \frac{n}{1-n}, \text{ and } 1 + \frac{1}{n} + \frac{1}{n^2} + \ldots = \frac{n}{n-1}.$$

He could also write

$$1 - 3 + 5 - 7 + \ldots = 0$$

and

$$\sin\phi - 2\sin 2\phi + 3\sin 3\phi - 4\sin 4\phi + \ldots = 0.$$

There is not much improvement here on the ideas of Leibniz and his contemporaries who believed that $1 - 1 + 1 - 1 + 1 - \ldots = \frac{1}{2}$, the series being derived from the expansion of $\frac{1}{1+x}$ when x is put equal to unity. Guido Grandi, who wrote the above series for $\frac{1}{2}$ is the form $(1-1)+(1-1)+\ldots$ concluded that

$$\tfrac{1}{2} = 0 + 0 + 0 + \ldots$$

and saw in this the mathematical analogue of the creation of the world out of nothing. But Euler cleared up the difficulty attending the logarithms of negative quantities. John Bernoulli had argued that since

$$(-a)^2 = (+a)^2 \therefore \log(-a)^2 = \log(+a)^2.$$

Hence

$$2\log(-a) = 2\log(+a) \text{ and } \log(-a) = \log(+a).$$

Bernoulli's fallacy is in the assumption that rules shown to be consistent for positive quantities may be applied without fresh investigation to negative quantities. Euler showed that a number has an infinite number of logarithms, all imaginary if the number is negative, and all but one imaginary if it is positive. Euler also made contributions to the calculus of finite differences, differential equations, continued factions, and the calculus of probabilities.

Besides covering this immense field in pure mathematics, Euler powerfully advanced the study of analytical mechanics. He gave the general equations for the motion of a body about a fixed point, and also for the free motion of a body. In hydrodynamics, also, he gave the general equations of motion. In astronomy he succeeded in giving approximate solutions to the 'problem of three bodies', a problem still unsolved in its general form. He wrote also on optics and on mathematical philosophy. Even this brief sketch will give some idea of Euler's immense activity. It has been calculated that his complete works would fill 16,000 quarto pages. One reason for this immense bulk of work is that Euler's method was to attack one special case after another, rather than to include all special cases in one general result. In this he was the exact opposite of Lagrange, the greatest mathematician of the eighteenth century, one of the very greatest mathematicians of all time, and a man whose capacity for generalization and abstraction has never been exceeded.

Joseph Louis Lagrange was born at Turin in 1736 and died at Paris in 1813. It was not until he was seventeen years of age that he showed any interest in mathematics. At that age he accidentally came across a memoir by Halley on algebra, and was immediately seized with enthusiasm for the subject. Without any assistance he became, in one year, an accomplished mathematician, and was appointed professor of mathematics at the royal military academy at Turin. A year later he wrote a letter to Euler containing the solution of an isoperimetrical problem which had been discussed by mathematicians for more than half a century. In order to solve the problem Lagrange invented the calculus of variations. Euler had himself been working at this calculus, but he saw at once the greater generality of the method used by Lagrange, and generously withheld his own paper on the subject in order that the young Italian should be able to claim the new calculus as his own. Lagrange's memoir placed him, at the age of nineteen, in the front rank of living mathematicians. From now until the age of twenty-six Lagrange worked incessantly, producing memoir after memoir of the highest quality. Many of these papers are concerned with dynamical problems, but they deal also with recurring series, probabilities, and the theory of numbers. For this last subject Lagrange had a quite peculiar gift, in addition to his more orthodox and important abilities. As a result of this activity Lagrange was acknowledged, by 1761, to be indisputably the greatest mathematician in the world. At the same time his doctors refused to be any longer responsible for his reason and even for his life unless he would rest from his tremendous labours. He took their advice, but it was too late for him to be restored to normal health. Henceforth he suffered constantly from profound melancholy. He did not write again till 1764, when he won the prize offered by the French Academy for the solution of the problem of the libration of the moon. He now made a visit to Paris and met the French mathematicians. In 1766 Frederick the Great wrote to him saying that 'the greatest King in Europe' wished to have 'the greatest mathematician in Europe' resident at his court. Lagrange accepted this invitation, and spent the next twenty years of his life in Berlin. Frederick lectured him frequently on his irregular manner of working, and finally Lagrange disciplined himself. Every night he set himself the next day's task and confined himself to that. He had the capacity of thinking out the whole of an investigation without putting pen to paper. When he had thought the matter out he wrote his memoir straight off without a single correction or erasure. It is quite impossible to summarize the work produced by Lagrange in those twenty years. He averaged about one memoir a month, some of which are long complete treatises, and he also wrote the magnificent *Mécanique analytique*, a work which has been called ' a scientific poem ' and which is, perhaps, the most beautiful mathematical treatise in existence. It was in this work that Lagrange introduced his generalized co-ordinates, in which are expressed his equations of motion. Lagrange was an extreme example of the analyst. He disliked geometry, and we are told that he was proud of being able to say in the Preface to his *Mécanique analytique* ' On ne trouvera point de figures dans cet ouvrage '.

In Lagrange's memoirs, which deal with every branch of mathematics, we find that the later ones introduce considerations of rigour. Yet even Lagrange was not fully aware of the importance of testing series for convergence, a fact which vitiates some of his theorems. In the later years of his life he was much occupied, amongst other things, with the foundations of the calculus, and in order to avoid Leibniz's infinitesimals and Newton's limiting ratios he endeavoured to base the whole calculus on algebra. In this he was unsuccessful, and he seems to have had doubts about the method himself. Towards the very end of his life he brooded over the foundations of Euclidean geometry. He composed a paper on the parallel axiom, and was just beginning to read it to the Academy, when he broke off with the remark, ' Il faut que j'y songe encore '.

Lagrange, like Newton, detested controversy, and he would let other people claim his discoveries as their own rather than argue the point. He was extremely timid in conversation, and his usual preface to any remarks he had to make was 'Je ne sais pas'.

Lagrange's great contemporary, *Pierre Simon Laplacc*, was born in Normandy in 1749 and died at Paris in 1827. He first became known, at the age of eighteen, by writing a letter to D'Alembert on the principles of mechanics. D'Alembert was so impressed that he secured the young Laplace the position of professor of mathematics at the *École Militaire* at Paris. Laplace now settled down to original research, and began his marvellous series of investigations in theoretical astronomy. Many of these are contained in his tremendous treatise, the *Mécanique céleste*, published in five volumes. It may be regarded as a great extension and translation of Newton's *Principia*. It is full of the most important theorems, and the mathematical ability shown is of the highest order. Every subsequent treatise on dynamical astronomy derives from it. One of Laplace's greatest investigations deals with the stability of the solar system. Newton doubted whether the system was mechanically stable, but Laplace showed that, if the planets are regarded as rigid bodies moving in a vacuum, their perturbations oscillate between certain limits, and equilibrium is preserved.

Many of the mathematical theorems used by Laplace were taken from Lagrange. Thus the idea of the potential originated with Lagrange, and Laplace made it the basis of his work on attractions, arriving at the famous equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

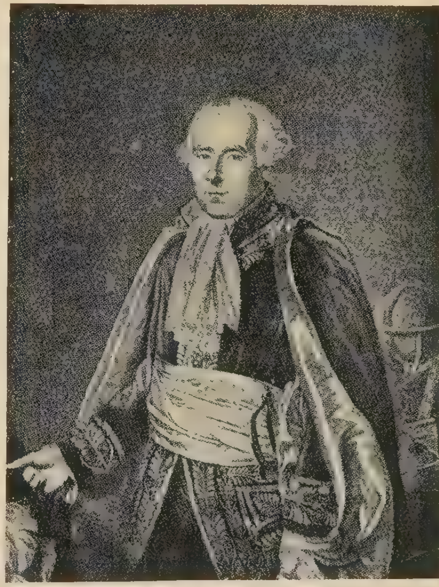
Laplace always seems to have been interested only in the definite problem he was discussing; he did not mind what methods he used, provided they led to a solution. As a result his work has none of the elegance of that of Lagrange, and is further unnecessarily obscure. Laplace's lack of interest in the actual analysis often led him to avoid the labour of writing it down, by substituting for it the remark 'Il est facile de voir'.

Besides writing the gigantic *Mécanique céleste* Laplace contributed to many branches of pure mathematics, and in particular to the theory of probabilities, for which he has done more than any other single writer. His treatise *Théorie analytique des probabilités* is, like the *Mécanique céleste*, very difficult to read, and the analysis he gives is not always perfect. But he had an almost infallible instinct for the right result. As De Morgan says of him, 'No one was more sure of giving the result of analytical processes correctly, and no one ever took so little care to point out the various small considerations on which correctness depends'.

But the period had now arrived when the use of comparatively lax methods of analysis was no longer to be permitted. In particular, the whole theory of infinite series was about to be overhauled. Gauss, Abel, Cauchy, were the leaders of this new movement, and with their work begins the modern period of mathematical rigour. It is reported that when Cauchy read his first paper on series Laplace hastened home and remained in seclusion until he had tested the convergency of all the series occurring in the *Mécanique céleste*. Fortunately he found them all to be convergent.

The early part of the nineteenth century was, indeed, a period of great critical activity. After the immense progress which had been made in developing the calculus and infinite series, mathematicians began to take stock, as it were, of the results that had been achieved. Many writers concerned themselves with the logical foundations of the calculus, but it took a long time before all difficulties were completely cleared up. More success attended the critical examination of infinite series, although it took many years before the new spirit, the spirit of mathematical rigour, affected all mathematicians. In geometry this spirit led to the creation of non-Euclidean geometry: we may mention Gauss, Johann Bolyai, and Lobatchewsky as being independently responsible for this important advance.

Other mathematicians of the eighteenth century would require mention in any history of the period, and even in this summary we may mention the name of *Legendre* (1752–1833), an analyst second only to Lagrange and Laplace, and famous chiefly for his work on elliptic integrals and the theory of numbers. And *Monge*, who reduced descriptive geometry to a science, must be mentioned as indicating that this period, so rich in analysis, is also of great importance in the history of geometry.



LAPLACE

But the chief difference between the mathematics of this period and the mathematics of our own time lies in the different conceptions of what constitutes a mathematical proof. A great deal of modern mathematics is the outcome, not of the merely formal development of what was known before, but of a profound critical examination of the foundations of the science. The super-structure, as it were, has been built up to ever greater and greater heights, but only on the basis of greatly deepened foundations. An account of modern mathematics, therefore, would be an account not only of technical mathematical processes, but also of the origin and development of mathematical philosophy.

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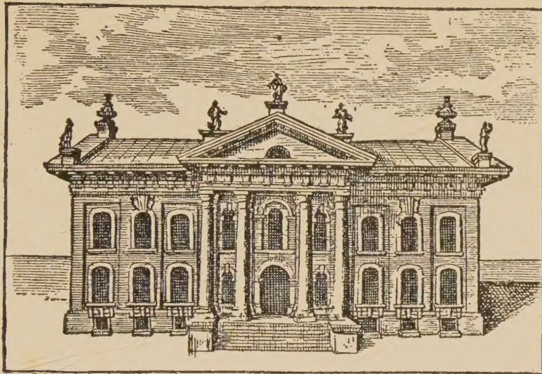

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