

Zdravka Paskaleva, Maya Alashka

BULGARIAN MATH 4

OR **8** B

TEXTBOOK

adapted for U.S. High Schools
and Excellent Middle Schools

Translated, Adapted, and Augmented
by

Zvezdelina Stankova

Founder and Director
of the Berkeley Math Circle

ΑΡΧΗ(Μ)ΕΔ
PUBLISHING HOUSE



Icons, Symbols, and Notation Used in the Textbook:

D	Definition	Basic fact/example	Write	Read
A	Axiom	In this lesson	$\Rightarrow; \Leftrightarrow$ LHS, RHS $\alpha, \beta, \gamma, \pi, \varphi, \psi$ Δ, Σ A', A_1 wrt	implies that; is equivalent to left-hand side, right-hand side alpha, beta, gamma, pi, phi, psi Delta/Sigma (= sum of) A -prime, A -sub-one with respect to
T	Theorem	Summary of lesson		
E	Elementary construction problems			
B	Basic construction problems			
	Clarification of the solution.		$A \in a, A \in a, a \in A$ $M \equiv N, M = N$ $M \neq N, M \neq N$ $OA \rightarrow, \angle AOB, \triangle ABC$ $a \times b = a \cap b = M$ $a \perp b; a \parallel b$	point A lies on line a M coincides with (is same as) N M does not coincide with N ray OA , angle AOB , triangle ABC lines a and b intersect in point M a and b are perpendicular/parallel
	Clarification of the basic facts.			
	Facts beyond the main material.			
	Historical notes and background.			
Exercises are grouped by difficulty:				
	Mandatory: basic proficiency.		$AV;$ $N, Z, Q, I, R; \emptyset$ $A \cup B, A \cap B$ $a \in A, A \ni a$	allowable values (domain) set of natural, integer, rational, irrational, real numbers; empty set union and intersection of sets a is in set A ; set A contains a
	Recommended: high proficiency.		h_a, m_a, l_a (to side a) s_{AB} P_F, p_F, S_F $\triangle ABC \cong \triangle A_1B_1C_1$ SAS, ASA, SSS, SSR	altitude, median, angle bisector perpendicular bisector of AB perimeter, half-perimeter, area of F $\triangle ABC$ is congruent to $\triangle A_1B_1C_1$ Δ congruence criteria: S = side, A = angle, R = right angle
	Elective: beyond the main material.		$k(O, r)$ $\xrightarrow{t}$ $r(O, \alpha)$ S_O, S_g	circle k with center O and radius r translation along vector $\vec{t}$ rotation with center O and angle α symmetry across point O /line g

BULGARIAN MATH 4 (8B)

Adapted for U.S. High Schools
and Excellent Middle Schools

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Foreword

No two people will agree entirely on all features of the “best” math curriculum, nor how it should be taught. Consequently, you need to read this introduction with an open mind, accepting that there will be differences of opinion and in approach, knowing that the “error” in how much the outlined philosophy differs from your views is a proof of the *seriousness of the situation with the U.S. math education.*

We all want to improve the math curriculum and the math experience, and the math proficiency, and the ultimate math success of our children. Do we want too much?

We want a **lot**: that is clear. And a **lot** has to be done in order to achieve this goal: a lot of sacrifices, hard work, and compromises between the views of various worlds. For the U.S. is a land where cultures mix (or don’t mix but co-exist), a land of opportunity, and a land of contradictions and extremes.

Our children come first: **before** our self-centered goals for self-improvement and well-being, **before** ourselves. Sacrificing personal comfort and personal self to the benefit of one’s students (or one’s children) is a sign of a great teacher (and a great parent).

When something powerful comes along and rings true, along with the listeners will come the frightened ones, who will be afraid to change, who will be scared to death to make the leap and try something new in the U.S.: something that has worked for a century elsewhere.

And such people will raise angry voices of distrust and will try to stamp out anything tender that has not yet taken root. But without such angry voices there will be no reality check, no feel for the seriousness of the situation.

The following introduction addresses a **typical** situation in a U.S. school. Exceptions do exist. There is no point in starting with this textbook series:

- unless you are ready to be completely *honest, goal-oriented, and selfless* as a teacher, a parent, or an administrator;
- unless you are ready – instead of trying to squash something new because it differs from what you are used to – to *embrace a philosophical viewpoint and a practical attitude that have proven tremendously successful elsewhere* and that have been *adapted to the U.S. reality and tested on a pilot school in the San Francisco Bay Area.*

Thus, we hope that you take this new math curriculum seriously and enjoy it too.

Mathematics is a powerful universal language. Everyone needs to and can own it to the extent of his/her abilities and potential. Let this new math curriculum be a big step for the students and community at your middle and high school towards reaching this goal.

Zvezdelina Stankova
Ph.D., Harvard University

Translator, Adapter, and Co-Author
Teaching Professor of Mathematics, UC Berkeley
Berkeley Math Circle Founder & Director
Deborah and Franklin Tepper Haimo Award
for Distinguished College/University Teachers
Princeton Review’s “300 Best Professors” in the U.S.

From the education battlefield:

“You have done a masterful job of adapting this curriculum to the U.S. reality.

The U.S. Education System needs this kind of prodding. The Common Core ideas are heading in the right direction but the instruments of change are too unwieldy and we cannot just stand by and wait to see what will happen.

I hope that your efforts will be successful. Thanks for all of your hard work.”

Tom Rike
Veteran math teacher
Oakland High School

Copyeditor of the Textbook Series

Reviews of the Program by Educators and Parents

Dr. Norm Prokup joined the faculty at College Preparatory School in Oakland, CA (the preeminent high school in the SF East Bay Area) in 2003. He has taught at the Berkeley Math Circle and the BMC Summer Program, and very notably, he has been the Geometry instructor for the 7A-8B groups at the MTRW Program. He graduated from Cornell University and received his Ph.D. from the University of Rhode Island at Kingston, studying nonlinear difference equations that model two-stage “healthy-infected” populations, suggested by epidemiologists. He has taught math ever since, working with students from grade 6 through college and covering a wide range of subjects, from algebra to ordinary differential equations.

“How powerful is the Bulgarian approach to algebra and geometry?”

It surpasses the traditional U.S. approach in two key ways:

1. Bulgarian Math presents algebra and geometry simultaneously over three years, guaranteeing much better retention than students can get from the traditional sequence: Pre-algebra, Algebra 1, Geometry, and Algebra 2.
2. The Bulgarian approach is more thorough. Although the authors didn't have the American Common Core standards in mind, the texts work exceptionally well with Common Core. Students who learn algebra and geometry from these texts develop superior skills in problem-solving, logic and reasoning, algebraic manipulation, and computation.

I grew up in northern Virginia. I had good teachers, many of whom cared enough about me to get angry when I wasn't doing my best. Growing up, math was my favorite subject, so I always started my math homework first. From engineering school, I learned the value of computation, precision, and fluency of execution. From mathematics, I learned that to really understand a subject, you must be able

to take it apart and then put it back together.

These values resound throughout the Bulgarian curriculum.

All my favorite math problems involve pictures, so I consider myself a geometer at heart.

- Solid geometry, vector geometry, plane transformations, and other topics that are often marginalized in an American curriculum play a central role in the Bulgarian texts. Hence, The Bulgarian curriculum is ideal training for future physicists, engineers, and computer scientists.
- Students emerge from this program ready to study analytic geometry and functions. Thus, this curriculum would be a superior choice for students from all grades 6 to 10.

I use the Bulgarian curriculum for my geometry classes at the “Math Taught the Right Way” Program at UCB. My students, who range from grade 7 to 10, have embraced this curriculum.

- I love teaching from these texts, and I love sharing them with other teachers.
- I eagerly await each chapter as it emerges, newly translated into English, and
- I'll be first in line to buy each new volume as it becomes available.

The miracle of the Bulgarian Math Program is that it works exceptionally well with a group of mature and industrious younger students. Indeed, the purpose of the Bulgarian texts is to provide the best possible training for tomorrow's mathematicians and scientists.

Dr. Norm Prokup

Elena Blanter received her Master's degree in mathematics from Saint Petersburg University in Russia. Elena's many years of experience in industry span mathematical risk modeling, statistical analysis, and development of financial analytical software. Elena is a co-founder of the Berkeley Math Circle Elementary Level (for grades 1-4) and has been teaching there since its inception in 2009. She is also a co-founder of the Stanford Math Circle Elementary Level at Stanford University, launched in 2011. Elena is thrilled to have the opportunity to bring her experience and passion to the Math Taught the Right Way Program (MTRW) at UC Berkeley.

When I taught a problem-solving class at MTRW in 2018, I was once again reminded of how ...

... fragmented and incomplete mathematics curriculum is in a typical American middle school. Even motivated and bright students do not have adequate mathematical language; they lose confidence when trying to solve a problem that is not given in a familiar format; they find themselves at a loss when presented with an atypical word problem, even when they have all the knowledge to solve it.

I grew up in the former Soviet Union. As a child, I remember the thrill of getting a new math textbook at the beginning of each school year and looking through it in anticipation of learning new and exciting mathematics. Every Sunday morning during my school years I took a bus across town to attend a math circle class. When my three children were growing up in the U.S., I wanted them to discover the beauty of mathematics. In their elementary school years, I supplemented their standard school curriculum with the Singapore Math textbooks available in the U.S., with several editions of Russian math textbooks, and with the Math Circle Elementary material.

Like learning a new language, becoming proficient in math takes time and practice. The earlier one starts, the easier it is and the greater proficiency one achieves. In a typical Eastern European mathematics program, middle school children are taught different areas of mathematics, starting in 6th grade and continuing through middle and high school. Such an approach supports giving children a solid mathematical foundation.

My children attended different public and private middle schools in the U.S. None of their math textbooks covered math subjects broadly and geometry was often absent from middle school math. I wish the Bulgarian middle school textbooks had been available at the time. This program covers many areas of mathematics continuously over the course of 3 years, including algebra, geometry, logic, reasoning, statistics, and problem solving. There is no one-year "crash" course in geometry or any other math discipline. The Program supports math knowledge acquisition and retention at a steady pace throughout all middle school years.

The exercises in textbooks are split into several levels to help with differentiation.

Math contest problems, interspersed throughout the book, can be assigned to advanced students.

Supplemental materials, offered as independent work, can add a "math circle" element and challenge even the most advanced students.

The Bulgarian program teaches children not just to look for the right formulas to plug in the numbers, or for the right template or a recipe to use; it helps teach students how to approach and solve a problem as a mathematician: "Let's pause to understand the problem, let's take a piece of paper and a pencil, and let's think".

I am excited to finally have an excellent middle school math program, and I love teaching it.

Elena Blanter

* * *

These textbooks emphasize the logical structures that seem to lack in the U.S. curriculum. Even starting in the 6A-B series, one can see the build up in knowledge - latter sections require results from previous ones and clearly state so - and the cohesiveness of the algebra and geometry, where the two subjects support each other. When used correctly, the books can explain most results in one lesson, using previous lessons. This structure helps students think mathematically. A true mathematician always asks "Why is this true?" when given a result. It is great to see this type of thinking exposed and instilled in students at such a young age.

In the U.S. math is commonly taught by memorizing algorithms without justifications: "It works because I told you so." And everyone knows that this does not sit well with a rebellious teenager. The essence and beauty of math is whether or not an answer to a question can be found. I was taught to question the results when I started 6th grade axiomatic geometry, using the translated French textbooks in Vietnam. And I am ecstatic to see that the present Bulgarian series provides students the same support and opportunity to question and develop their understanding of mathematics.

Harry Mai-Luu, Math Major at UCB
BMC/MTRW assistant and instructor

Here follow reviews of the Bulgarian Math curriculum by two more instructors, Elysée W.-Egolf and Ellen Kulinsky, at the 6th grade Algebra, Geometry, and Problem Solving classes at the “Math Taught the Right Way” (MTRW) Program at UC Berkeley.

I grew up in the U.S. with textbooks ...

... that were rote and unmotivated, with no applications for algebra students, no geometry or number theory for 6th graders, and no guidelines in writing mathematics. The curricula that are popular in the Bay Area at the time of writing have sought to resolve this issue by making the entirety of student education exploratory, without practice or summarization of what is expected - and in both paradigms, the textbook authors take dozens of pages to get to the point.

The current adapted Archimedes curriculum, by contrast, gives examples for students to explore and then guides them to correct conclusions. Algorithms are offered only after clarifying their logic and students are taught to work clearly and thoroughly through both numerical exercises and problems requiring more complex reasoning - and all this while calmly weaning reliance from calculators. It picks up where books like Singapore Math leave off, and offers students of all backgrounds and abilities entry into the deeper understanding they require to master future material.

Because the topics are covered so thoroughly and a workbook is included with extra problems, there is absolutely no need for teachers to create extra worksheets, which means that the teacher can focus on planning lessons and evaluating student work. The Bulgarian Math textbook is more relaxing for both student and teacher to work from.

Elysée Wilson-Egolf

BA in Mathematics, UC Berkeley

Bentley High School Math Teacher
Instructor at MTRW and BMC

These texts are such an improvement from ...

... the typical textbooks used in the American math education system. Rather than merely presenting formulas, these materials focus on their derivation and various applications. The result is remarkable, studying math becomes an engaging exploration for the students, rather than the tedious memorization and monotonous plug and chug students may be used to in the American math curriculum.

The books contain beautiful explanations and plenty of examples, easing students into challenging problems. And although the books cover a wide range of mathematical subfields (geometry, algebra, etc.) the students are able to see how concepts from different subfields are integrated, as they are in the real world, outside the classroom.

In my personal experience, growing up, I attended a Math Circle, which exposed me to creative, thought-provoking problem-solving approaches, as well as the connections between mathematical subfields, so I knew that the math classes I was taking in middle and high school were not a fair representation of how enjoyable studying math could be. These are the first math books I have come across that bridge the gap and incorporate the depth, clarity, and excitement of a Math Circle, while also tailored to the American math standards.

These materials are the perfect tool for students and math teachers alike to truly experience and share the beauty and joy of studying mathematics.

Ellen Kulinsky

Math Major, UC Berkeley
Instructor at MTRW and BMC

The quotes on this page are from more pilot teachers, parents, and educators. You will come across them in the Introduction chapter, in their original context. Thus, we strongly encourage you to carefully read the Introduction first, before continuing with the actual math material.

"This book has been a tremendous resource for my students, and I wish I had it when I was in school. The lessons are laid out in a very clear manner. The math is approachable or challenging to meet the needs of a child at any level. There are many math books that claim to reach every person. This book successfully does so, and it is one of the best."

Natalya St. Clair, pilot teacher
co-author, "The Art of Mental Calculation"

"I feel that the Bulgarian texts are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there been this year also a translated 7th grade Bulgarian textbook I could use, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, pilot teacher

"The Bulgarian curriculum is more rigorous than others I have used as it introduces Algebra in early grades. Variables are integrated into daily lessons so that students become as familiar with manipulating equations with x and y as they do with real numbers. This will pave the way for an easy transition to Algebra and set them up for success in higher grades."

Laurie Gold, pilot teacher

"Our daughter has ... learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems."

This program keeps her interested and excited."

Eve Maidenberg

6th grade pilot parent

"Thank you so much... for this feedback! Our daughter is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"

6th grade pilot parent

"As a teacher and teacher educator, I am familiar with several ...

... math programs that have been used throughout the past 20 years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.

Teaching is sort of like farming. You never know how your crops are going to turn out, but I have noticed with my son that his work has improved, and I really like the way he is talking about math. I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics."

Jacqueline Regev, 6th grade pilot parent and educator

"This Middle and High School Math Program is deeply aligned with the Common Core Mathematics Standards".

Dr. Phil Daro

One of the Three Principal Writers
of the Common Core
Mathematics Standards

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Introduction to the Program

1. Mission Impossible

Every year, for the past 20 years, while working* with 250 students in grades 5-12 at the Berkeley Math Circle (BMC), I have been asked over and over again by parents, students, and teachers alike:

*"Can you give us good textbooks for our middle and high school math curriculum?
What we have is not working and it hasn't worked for years!"*

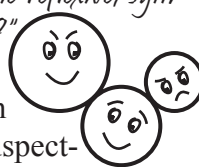


BERKELEY
MATH CIRCLE

And for these 20 years I could think of only two sources that I would also give to my own children: my *old Bulgarian middle school textbooks*, and the *Art of Problem Solving books (AoPS)*. But **neither of these choices would have worked**, despite the fact that they are, in my opinion, among the best middle school textbooks in the world. Indeed, they are written for *advanced* audiences as measured by U.S. standards:

The very first sentence in my old *Bulgarian 6th Grade Algebra* textbook would stump many of my freshman college students: "Since we have worked [in 5th grade] with the set of rational numbers $\mathbb{Q}$ and are familiar with its properties, we move on." And without further ado, Lesson 9 went on to studying the subtle differences between functions and relations!

My old *Bulgarian 6th Grade Geometry* textbook did not lag behind. On page 42 it asked: "Is the relation 'is tangent to' on the set of circles in the plane reflexive, symmetric, or transitive?" Just to explain what the question means to an unsuspecting, typical 6th grader would take a whole lot of time!



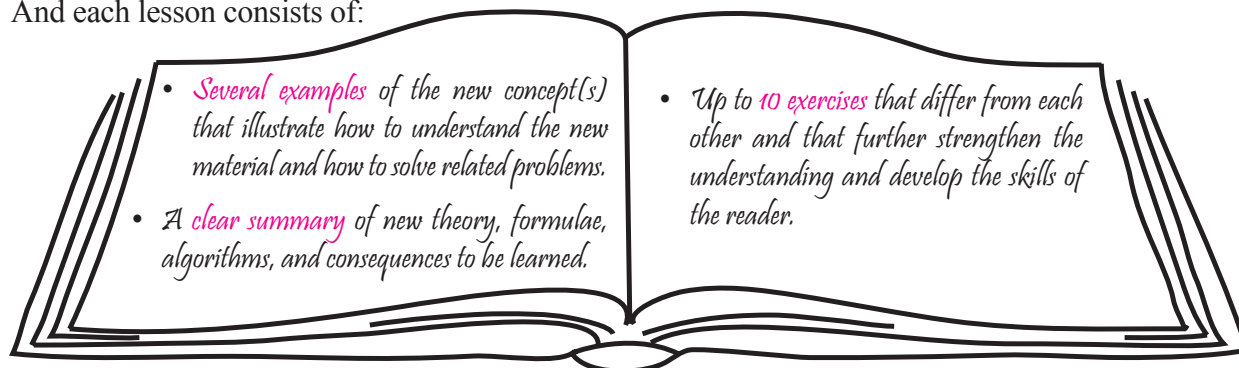
Although the *AoPS books* are a wonderful resource, they are not designed as "text-books" in a regular math class environment. I tend to recommend these books for my more advanced Math Circle students, who are guaranteed to have extra support outside of a typical school classroom.

And so, short of designing a new program myself, I was helpless to help anyone seeking a better middle school math curriculum....

2. The Find

....That is, until I asked my mom to send me some recent Bulgarian math textbooks, to teach my own kids in the U.S. She quickly found the most popular textbooks in Bulgaria: the present series from "Archimedes." When I opened them, I was immediately taken by their main feature: *Each lesson is on exactly 2 pages!*

And each lesson consists of:



* I am a Professor of Mathematics at UC Berkeley. I am originally from Bulgaria, which has a rigorous mathematics curriculum. I founded BMC to help pass the math enthusiasm on to the younger generation in the San Francisco Bay Area.

The textbooks are written in a *straightforward* and *simple manner*, capturing the *elegance* and *robustness of mathematics* itself. There is no “hide-and-seek”:

- everything is *crystal clear*;
- *one look* at a lesson suffices to see what it will teach you;
- there is no need to look around on the Internet for other sources: the *textbook*, *workbook*, and *problem collection* are all one needs for a *successful course* in middle school mathematics.



A pilot 6th grade teacher shares:

"I feel that the Bulgarian textbooks are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there also been a translated 7th grade Bulgarian textbook I could use this year, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, teacher

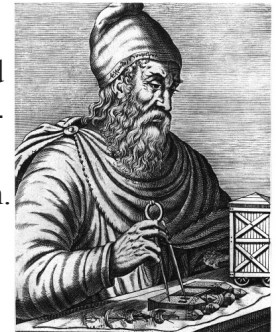
3. Origins

This middle school math program is based on one of the leading programs of today in Eastern Europe, developed and implemented by the well-known Bulgarian mathematician and educator Professor Georgi Paskalev. His relatively new mathematical program specifically targeting middle and high schools in Bulgaria has conquered 60-70% of the Bulgarian education market during the single decade since its conception.

The new program:

- has *partially departed* from the traditional “old school” Bulgarian program, and
- has *crossed boundaries* to meet the standards and demands of a global educational market, such as in the U.S.; yet,
- has *preserved the intrinsic pedagogical and mathematical values* of the old curriculum.

Quite appropriately, the company founded by Professor Paskalev is called “*Archimedes*”: its publications live up to the name of the great Greek mathematician, scientist, and inventor from antiquity.*

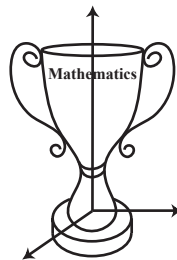


4. Purpose

The present math program extends the mathematics taught in elementary school and completes it to a comprehensive K-8 math program that prepares students:

- to enter any high school with *confidence* in their math skills and knowledge;

- to be *successful* at any level of high school math classes;



- to develop *maturity* and a *life-long relationship* with mathematics that:
 - will distinguish them among their peers;
 - will serve them well in any profession and in any endeavor in which they chose to engage.

* Archimedes' picture is from Wikipedia: <https://la.wikipedia.org/wiki/Archimedes#/media/File:Archimedes1.jpg>.

5. The Temple of Mathematics

The math education of every child is like a *building*; every year we hope that a new floor emerges on top of the previous floor. Unfortunately, more often than not, our children come *under damaging* (even though, perhaps, well-intended) *actions of math educators on any floor of this building*. For example,

- What happens when a teacher decides to spend 3 months reviewing single-digit multiplication in 4th grade?

Well, the 4th graders become really skilled with their multiplication tables, and, say,

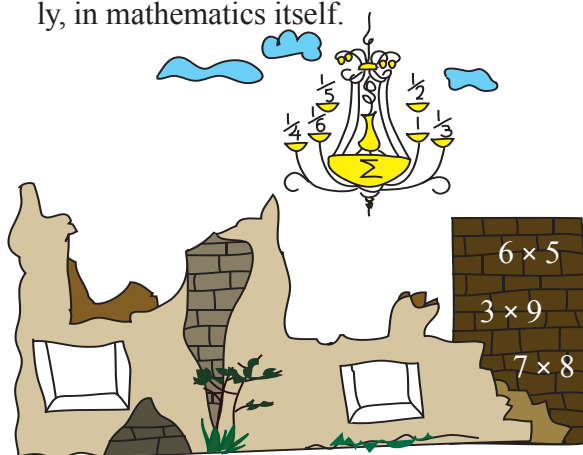
- a 6-foot thick wall of “single-digit multiplication” is erected on the 4th floor of the building.

Alas, time is not stretchable, and such educational decisions inevitably cause:

- some other wall on that same floor to be thin or non-existent; e.g., a “wall” of word problems, logical thinking, or a rich geometric vocabulary.

As a result, when in 5th grade the next teacher tries to build “walls” upon such previously expected but non-existent or weak walls, not surprisingly,

- the structure collapses! And in the ensuing debris, time is wasted figuring out what and why it happened. Worse, a student may lose confidence in his/her math skills, in the teacher’s abilities, in the integrity of the math program, and, sadly but not infrequently, in mathematics itself.



What may strike us as even *more dangerous* for the young developing mind (yet, still well-intended) is the desire of many textbooks and some teachers nowadays (perhaps, under pressure from school administration or for other reasons) to occasionally show off via:

- *extra-curricular activities* that are unsupported by and incompatible with the math building erected up to that moment. A typical example is an “enrichment” section on:
 - *Series!* This topic, ordinarily covered in a college Calculus course, has made its debut in standard, well-known U.S. 6th grade textbooks, alongside computer codes that intend to demonstrate the “growth” of the “partial sums of the series”.

Let us reason logically here.

How can one *truly understand* why the sum

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$$

adds up to infinity, but the sum

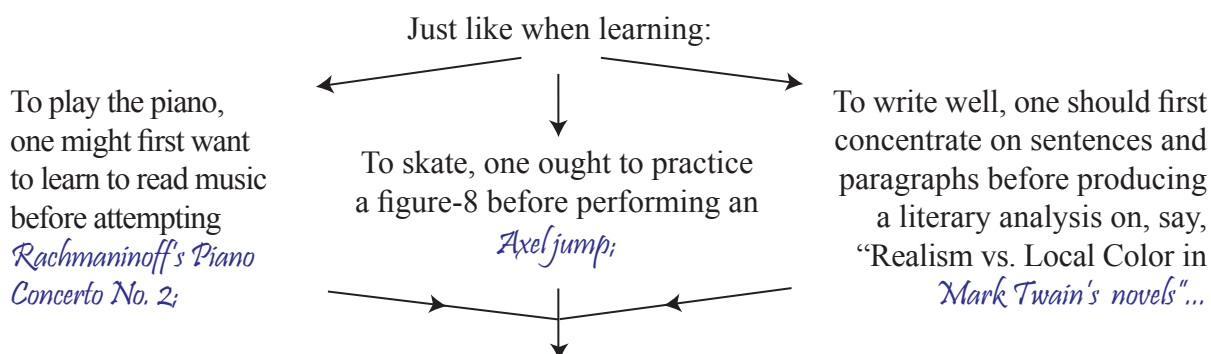
$$1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots$$

does not, *before* having mastered, say, the *distributivity laws*?! Although U.S. students may be able to recite from memory these distributive laws (e.g., $a \cdot (b + c) = a \cdot b + a \cdot c$), by and large, they will have no clue when, why, and how to apply these laws, so much more efficiently.

Thus, we ask again: what effect will such a “fancy” and untimely introduction of high-level math ideas have on the budding young mind? To put it bluntly, the situation will be equivalent to hanging a fancy chandelier on a non-existent ceiling. Imagine the “aftermath” of such an action, the loss of time, energy, and confidence.

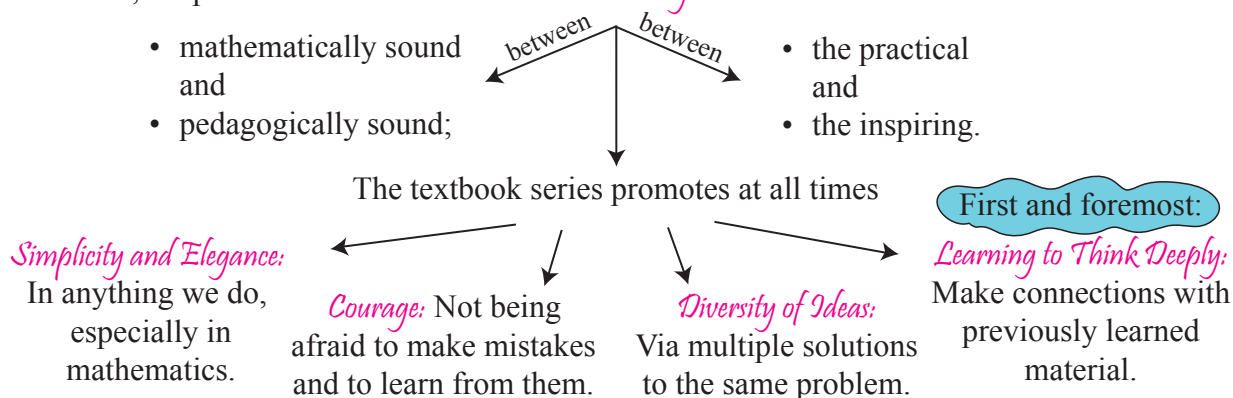
And we are **not** talking about the precocious youngster who can handle with ease and excitement any bombardment with advanced math material, whether on series, in combinatorics, abstract algebra, or other such topics! We have plenty of such young people at the Berkeley Math Circle and observe this phenomenon on a weekly basis all the time.

- We are talking about the *typical* U.S. student, whose primary or only source of learning math is through math classes in school. →
- The math structure for this student will be *jeopardized* by such disorderly and out-of-context teaching.



... So there should be *order, balance, and harmony* in everything we do, teach, and learn. We **cannot** lower our standards when teaching mathematics to young minds. On the contrary, we must be especially careful because of the *hierarchical structure of mathematics*.

And thus, the present textbook series* has found the *golden ratios*:



These are not just catchy, “politically correct” phrases intended to sell more copies of this textbook. They truthfully describe this math program and help build a solid math structure, with adjoining “walls, windows, and doors,” through which to move from topic to topic and make connections between various concepts, formulas, and ideas.

For someone who has not worked deeply with mathematics before, the words *love, creativity, and excitement* are usually light years away from having anything to do with math. Yet, here they all are, in a testimony of a pilot 6th grade parent, leading us to the next topic – *how a student interacts with and feels about mathematics*.

* The pun was not intended in the former, but it was intended in the latter instance! ☺

A pilot 6th grade parent shares:

“Our daughter has learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems. This curriculum keeps her interested and excited.”

Eve Maidenberg, parent

6. Relationship with Mathematics

With some notable exceptions, especially in Geometry, the actual math content of these textbooks does not “look” much different from just about any standard U.S. middle school textbook. After all,

A *linear equation* $5x - 7 = 11$ looks just like a linear equation, no matter if it is displayed in a U.S., Uruguayan, or Bulgarian textbook.

The *Pythagorean Theorem* will still say $a^2 + b^2 = c^2$ for the two legs and the hypotenuse of a right triangle.

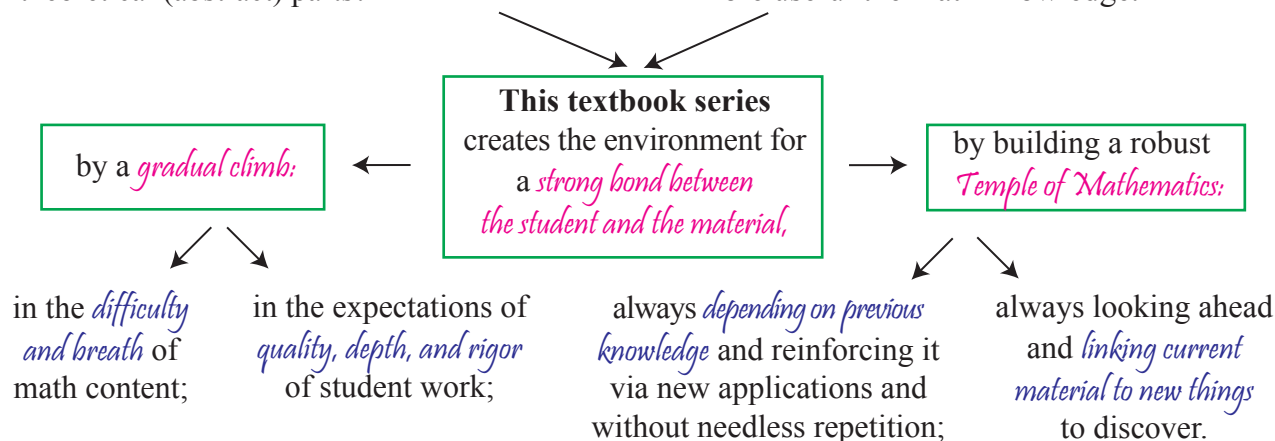
A shortcut to finding out if a number can be *divided by 3* (without remainder) will still be to check if the sum of its digits is itself divisible by 3.

... because *mathematical truth is universal*.

Beyond the content, two *fundamental questions* that a good teacher must ask about any textbook are:

Question 1: How are the math facts *presented*? In what order and what style? Is there a balance between computational (algorithmic) parts and theoretical (abstract) parts?

Question 2: What *relationship* between the student and the material is developed? The deeper the relationship, the more lasting and more useful the math knowledge.



At any time throughout the school year the place and importance of the concepts, ideas, and techniques studied are *clear within the "bigger picture" of Mathematics*. Unfortunately, just as described on the previous pages, meddling with textbooks and curricula is a wide-spread phenomenon in the U.S. middle school math education! Here are four sure ways to ruin the structure and purpose of any math textbook, whether good or bad:

1. Omitting sections.
2. Changing the order of the material in the textbook.
3. Cluttering lessons with hand-outs from other curricula.
4. Sacrificing depth and teaching to the answers!...



Remember that no one wants to climb up a flimsy structure! As a result, over time the top floors of the Math Temple will become uninhabitable if the actions in the red box run amok in math lessons.

Instead, the deep thought and math wisdom put into creating these textbooks must be respected, or there is no point in starting with this program. Thus, the present textbooks must be *followed just as systematically* as one studies a new language, violin, or chess.

7. A Cultural Shift

Several features of the textbooks, which are typically ignored or de-emphasized in standard U.S. middle school curricula, represent a *cultural shift* in how mathematics is viewed and studied in the U.S., but which are very well-known around the world and have been adopted by many countries for decades, if not centuries.

7.1. Reading Mathematics. Every math problem has “words” – these are concepts expressed in different ways, whether as numbers, standard everyday words, or by visual aids such as diagrams and figures. Emphasizing the different forms of “words” as a regular part of math language and math communication and *learning to read, interpret, and write in “words”* is a major part of the new curriculum. The pilot parent’s comments below addresses this and more:

A pilot 6th grade parent shares, 7 months after the new curriculum was launched:

“As a teacher and teacher educator, I am familiar with several math programs that have been used throughout the past twenty years in California. The Bulgarian math program is exceptional in the way it supports students’ deep understanding of mathematical concepts, develops students’ ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.”

I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics.

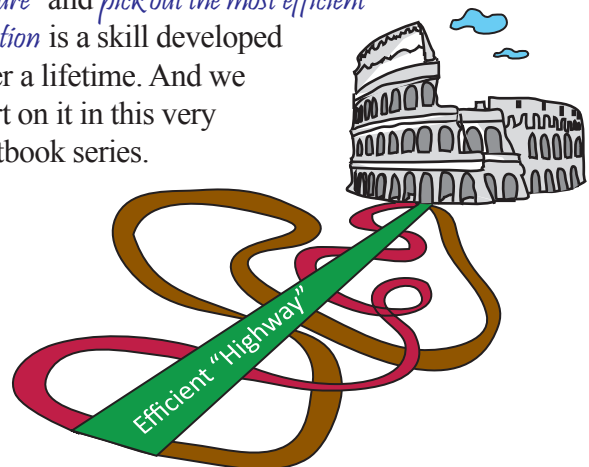
Jacqueline Regev, parent and educator

7.2. Logic and Communication. Learning to know *correct from flawed reasoning* and being *able to explain* one’s mathematical ideas smoothly and convincingly to others will represent a gradual move towards *a more mature relationship with mathematics*.

7.3. Writing Mathematics. “Showing your work” is only the beginning to understanding how math really works. All problems have answers in the back of the textbook. Bringing back just an answer on a homework problem from the textbook will be worth no credit. *The explanation that leads to the answer* will be what counts in our study of middle school mathematics. Even harder than learning to correctly interpret problems is learning to consistently *write solutions in a correct, complete, and clear way*. (See Lesson 5 in the 6A textbook.)

7.4. Multiple Solutions. Although there is usually (but not always!) only *one* correct answer to a math problem, the beauty of mathematics is that there may be *different solutions leading to that answer*. These are not “subjective opinions”; rather, they are objective mathematical creations that obey the laws of Logic. Students will learn that each solution (and even each incorrect attempt) has its value and usefulness in the long run and that one needs to *open up to others’ ideas and ways of thinking* as a path to enriching oneself, to becoming more proficient and, ultimately, wiser.

7.5. Efficient Solutions. Yet, among “all the roads [that] lead to Rome,” there might be a shortest or an easiest to follow. The ability to *see “the big picture”* and *pick out the most efficient solution* is a skill developed over a lifetime. And we start on it in this very textbook series.



8. The Pillars of Mathematics

There are *three pillars* of mathematics: Algebra, Geometry, and Logic/Problem Solving. They must be present at any stage of pre-college math education.

8.1. Algebra*

There is so much fuss to distinguish between Pre-Algebra, Algebra 1, and Algebra 2... when, in fact, there is no such thing as “Pre-Algebra” as a math subject, nor are there “Algebra 1 or 2”! In a well-designed and well-executed K-12 program,

- *Arithmetic*
↓ gradually turns into
- *Algebra* $\xrightarrow{\text{naturally leads to}}$ • *Real Analysis.* **

Alas, the typical U.S. student:

- hops along uneven and vaguely defined passages between Pre-Algebra, Algebra 1, Algebra 2, Pre-Calculus, and Calculus;
- experiences a chop-chop approach of poorly matched Algebra curricula from grade to grade;
- is taught to be more concerned about memorizing algorithms*** and passing test benchmarks,

than what is truly important:

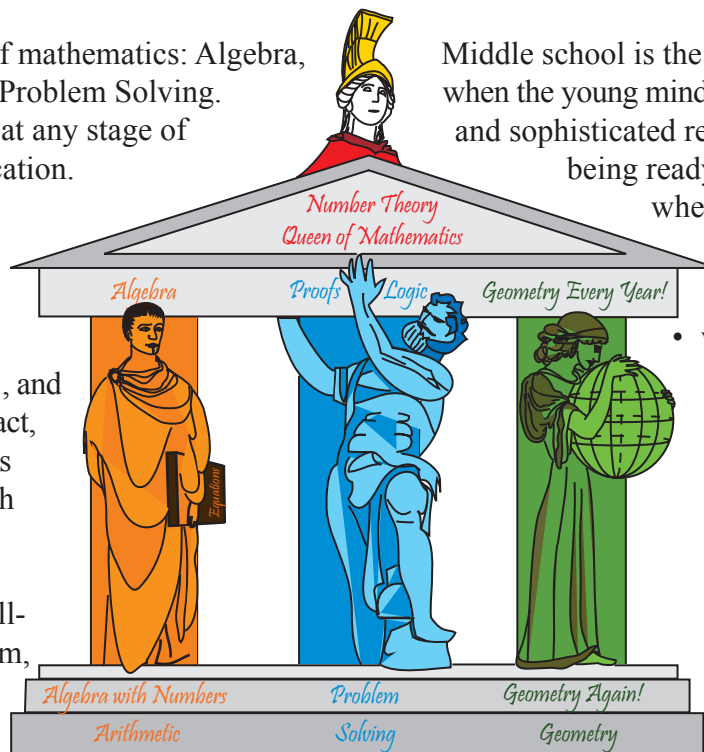
- *making connections* and fitting the many pieces together in a giant “jig-saw puzzle” to see the “*big picture*”;

and little by little,

- building a *solid Pillar of Algebra* in his/her Temple of Mathematics.

Middle school is the time when math clicks: when the young mind can handle logical leaps and sophisticated reasoning. Yet, instead of being ready for the “big moment,” when middle school knocks on the door, teachers and parents alike start:

- worrying about students “*missing algebra skills*”;
- pulling the class back to arithmetic operations because all of a sudden “*pre-algebra*” is too hard;
 - going in circles;
 - losing time, energy, and confidence; and
- ultimately, *weakening the math structure.*



8.2. The Missing Algebraic Link

Why is this happening? The major Algebra mishaps in U.S. middle schools are due to events that have already occurred in elementary school, and are NOT necessarily due to poor arithmetic skills. Expecting that a middle school student will leap from *arithmetic operations with numbers*, e.g.,

$$97 \cdot 2016 = 195,552,$$

to *algebraic operations with symbols*, e.g.,

$$(a + b)(a - b) = a^2 - b^2,$$

is asking for trouble. There is a whole array of missing intermediate steps, to which we refer as “*Algebra with Numbers*” and which can and should be started very early in the learning process.

How early? On the next page we list some examples from *early grades in Bulgaria*. They also represent well what happens in the algebraic progression in other countries around the world.

* Algebra was present in ancient Greek mathematics mostly as *geometrical algebra*. The word “algebra” is Latin for the Arabic “al-jabr,” or “the reunion of broken parts”. In our “divine” interpretation of mathematics, the Greek mathematician *Diophantus, the Father of Algebra*, is supporting the math structure on the left and is the only non-deity in the drawing; yet, as a mortal, he had tremendous world influence in both Algebra and Number Theory.

** a.k.a. *Calculus* in first 2 years of U.S. college education.

*** Automated “recipes” for some types of problems.

Sample “Algebra with Numbers” Problems from Elementary School in Bulgaria

1st grade. Calculate in two ways $7 + 9 + 3$.

- **Expected written solution:**

$$(7 + 9) + 3 = 16 + 3 = 19;$$

$$(7 + 9 + 3) = 10 + 9 = 19.$$

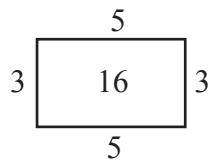
2nd grade. Fill in the blanks. Where have we seen such calculations?

$$[5 + 5] + [3 + 3] = 2 \cdot \diamond + 2 \cdot \diamond = \diamond$$

- **Expected written solution:**

$$[5 + 5] + [3 + 3] = 2 \cdot 5 + 2 \cdot 3 = 10 + 6 = 16.$$

I remember! The sides of a rectangle that are opposite are equal. The perimeter of the rectangle is twice one of the sides plus twice the other side.



3rd grade. Calculate the expression:

$$1000 - (132 \div 3) \cdot 4.$$

- **Expected written solution:**

$$\begin{aligned} 1000 - (132 \div 3) \cdot 4 \\ = 1000 - 44 \cdot 4 = 1000 - 176 = 824. \end{aligned}$$

4th grade. Check whether the equality

$$(a + b) - c = a + (b - c)$$

is true for $a = 347$, $b = 217$, and $c = 105$.

- **Expected written solution:**

$$\begin{aligned} (a + b) - c &= (347 + 217) - 105 \\ &= 564 - 105 = 459; \end{aligned}$$

$$\begin{aligned} a + (b - c) &= 347 + (217 - 105) \\ &= 347 + 112 = 459. \end{aligned}$$

Since we got the same answer 459, the equality is true in this case.

5th grade. Find the unknown number x if

$$8\frac{4}{5} \div x + 5.3 = 25 \cdot 0.3.$$

- **Expected written solution:**

$$\begin{array}{l|l} 8.8 \div x + 5.3 = 7.5 & 8.8 \div x = 2.2 \\ 8.8 \div x = 7.5 - 5.3 & x = 8.8 \div 2.2 = 4. \end{array}$$

This sequence of five random exercises in “Algebra with Numbers” starts in 1st grade in Bulgaria and continues all the way to 6th grade. And then *the “true” Algebra commences*, without hiccups, without drama, and with huge success, because:

- students have been *preparing for Algebra for a long time*, even if no one called it “Algebra”;
- a *continuum in studying Algebra* has provided smooth transitions at every step of the way.

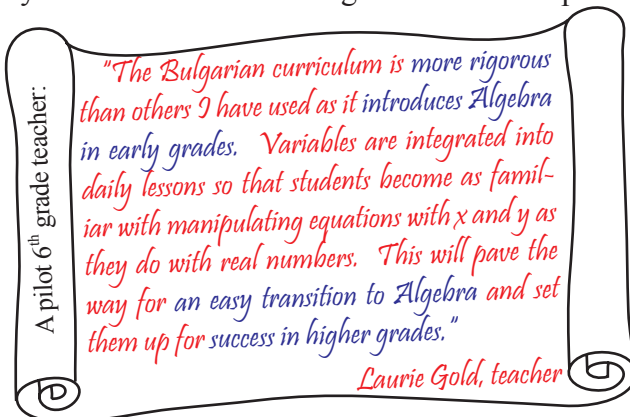
And just as playing the piano for the 6th year will likely encompass some harder, lovelier pieces, so will studying Algebra for the 6th year conquer tougher, nicer problems.

This short list of problems, culminating in the 5th-grade equation with fractions and decimals, also explains why we could NOT start our 6th graders in the U.S. on the 6th grade Bulgarian curriculum: they were NOT ready for it! We roughly estimated that they were lagging behind their Bulgarian peers by half a year in terms of algorithmic manipulations; and they were at least a year behind in algebraic skills and math maturity.

Of course, the easiest thing was to give up. But anyone could do that. We could do better. We did MUCH better! In fall 2015 we started the new curriculum in the pilot middle school:

- by first covering all of the Bulgarian 5th grade curriculum in 6th grade here; and
- only then moving to the Bulgarian 6th grade curriculum in the last third of the 6th grade here and covering half of it.

“Were you successful with the Algebra part?” you ask. Let one of the 6th grade teachers respond:



8.3. The Geometry Neverland



In our drawing of the pillars of mathematics Geometry is represented by Gaia, the Greek Goddess of Earth (“geo” = Earth in Greek). The truth is that Geometry is the “*Cinderella*,” with U.S. math education being its “step-mom.” In the K-8 grades Geometry appears sporadically here and there, with few (if any) connections to the rest of the math curriculum.

Later, in U.S. high school curriculum, there is *one intense year of Geometry*, during which:

No other math courses are studied. This in itself is a recipe for trouble: the few connections between Geometry and the rest of mathematics that were hinted before in the math curriculum are mostly forgotten during that year.

Geometry is studied in isolation!

4 years worth of Geometry are compressed in one year and they are presented to, by and large, students unprepared for such a huge single “dose” of Geometry.

Unlike making bread, there is no time for Geometry knowledge to “rise”!

Proofs are introduced through this Geometry course, often in the notorious “2-column format.” Truly powerful Geometry theorems are not reached. As a result, many students not only do **not** learn how to do proofs, but also...

They end up, by association, disliking Geometry itself!

Question: Why talk about a **high** school Geometry course in the midst of a **middle** school curriculum?

Because everything starts at an early age and builds up over the years, regardless of whether it is Algebra, Geometry, Logic, playing an instrument, or learning to write excellent essays. And if we do not have the *vision of what awaits the young learner* in a few years from now, we will be doing a big disservice to the young learner, and we will be cutting off his/her chances to succeed in the future.

Knowledge and skills in Geometry cannot be amassed “miraculously” in a year. *Geometric intuition develops over time*, not instantaneously. To package and present several years worth of Geometry in one course to the untrained mind, and to top it off with introducing formal proofs (a hard topic even in a college curriculum) inevitably leads to a poor outcome.

Worst of all, by the time the high school Geometry course comes, *“the train is already gone”*: it is too late to repair the damage that was systematically done over the K-8 years to the Geometry Pillar in the Temple of Mathematics. Consequently, the young budding math mind has *missed the creative and formative time* in middle school for a fundamental geometric development.

Insufficient Geometry knowledge and skills will impede performance in Algebra and will reverberate throughout a person’s whole life, including college and post-college career.

The truth again:

- Geometry has been sacrificed* in favor of
- Arithmetic in elementary school;
- Algebra in middle school, and
- Pre-Calculus/Calculus in high school.

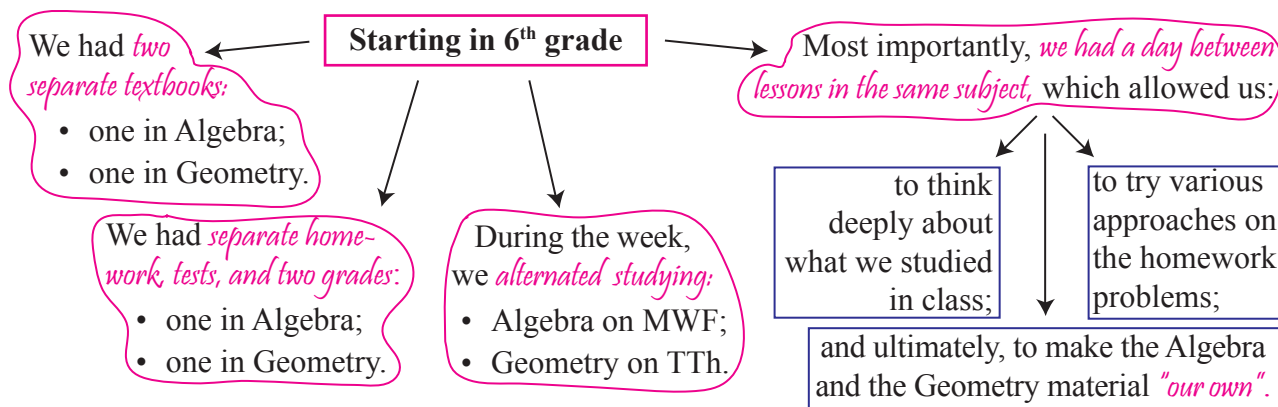
The Geometry Pillar will never be built with the strength and height as it should be in order to hold up a robust, elegant, and long-lasting Temple of Mathematics.

The lack of serious and consistent Geometry in early grades makes it impossible to overturn the “algebraic dominance” in the K-12 curriculum. Can this be changed? Can Geometry “Neverland” be replaced by *a healthy symbiosis between Geometry and the rest of mathematics*? Let us look at what could have been.

8.4. A Personal Lamentation

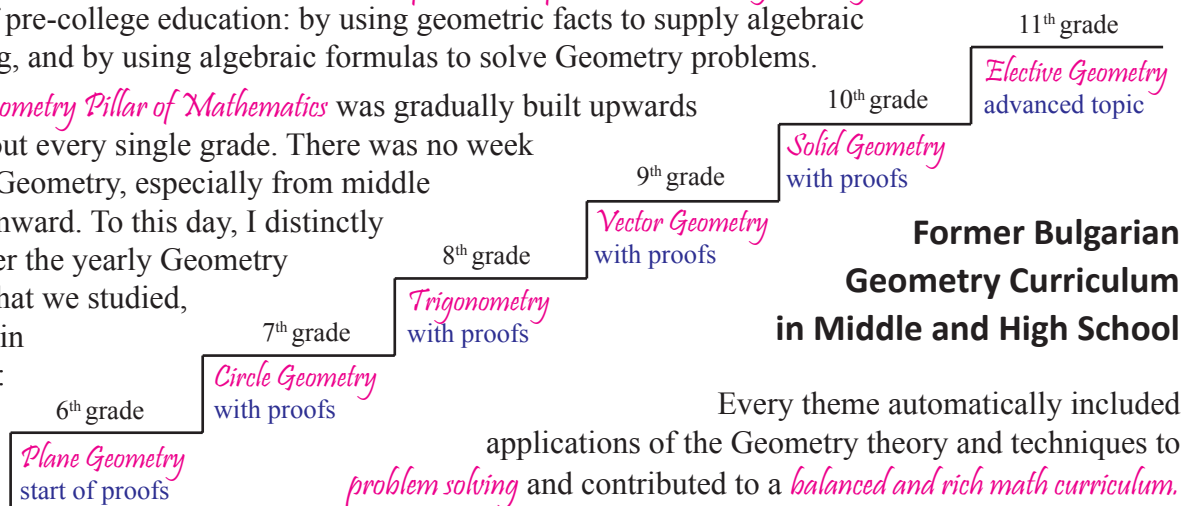
Or ... the Golden Geometry Days of My Childhood

From my 25-years of experience teaching just about every undergraduate math course in college, the most mystifying formulas to my students, the least well understood, and (it goes without saying) the least frequently proven in pre-college years are the *trigonometric* ones. During the socialist times *in Bulgaria*, trigonometry was part of the standard 8th grade Geometry curriculum; the theory behind such formulas was systematically studied, and the formulas themselves were rigorously proven via geometric or, as needed, algebraic methods and applied to various problems for years in a row. In fact,



Although in two formally separate subjects, our studies of Algebra and Geometry were synchronized and mutually enriching. *A constant and deep relationship between Geometry and Algebra* was reinforced at all stages of pre-college education: by using geometric facts to supply algebraic reasoning, and by using algebraic formulas to solve Geometry problems.

The Geometry Pillar of Mathematics was gradually built upwards throughout every single grade. There was no week without Geometry, especially from middle school onward. To this day, I distinctly remember the yearly Geometry themes that we studied, starting in 6th grade:



Every theme automatically included applications of the Geometry theory and techniques to *problem solving* and contributed to a *balanced and rich math curriculum*.

This is the *ideal symbiotic model of Geo-Algebra* that I would like for my children and grandchildren.

Unfortunately, high school math curriculum in the U.S. is driven, by and large, by college requirements and, let's face it, earning "bonus entrance points" to the more competitive universities. Hence, it is ultimately a *Calculus-driven curriculum*, regardless of whether students complete Calculus courses in high school, or not.

In turn, middle school math curriculum aligns itself with high school priorities. Backtracking, the pre-requisite to Calculus is Pre-Calculus, and the pre-requisite to Pre-Calculus is Algebra. Thus, Algebra "wins" by default, due to an established *Algebra-driven* and *Algebra-dominated environment*.

8.5. A Reality Check: Geo-Algebra in the U.S.?!

It is unrealistic (although not entirely impossible) to imagine that the pre-college curriculum in the U.S. can quickly change so much that it will embrace Geometry as an equal counterpart of Algebra, will split textbooks and classes into Algebra and Geometry ones, and will employ *the Geo-Algebra symbiotic model*.

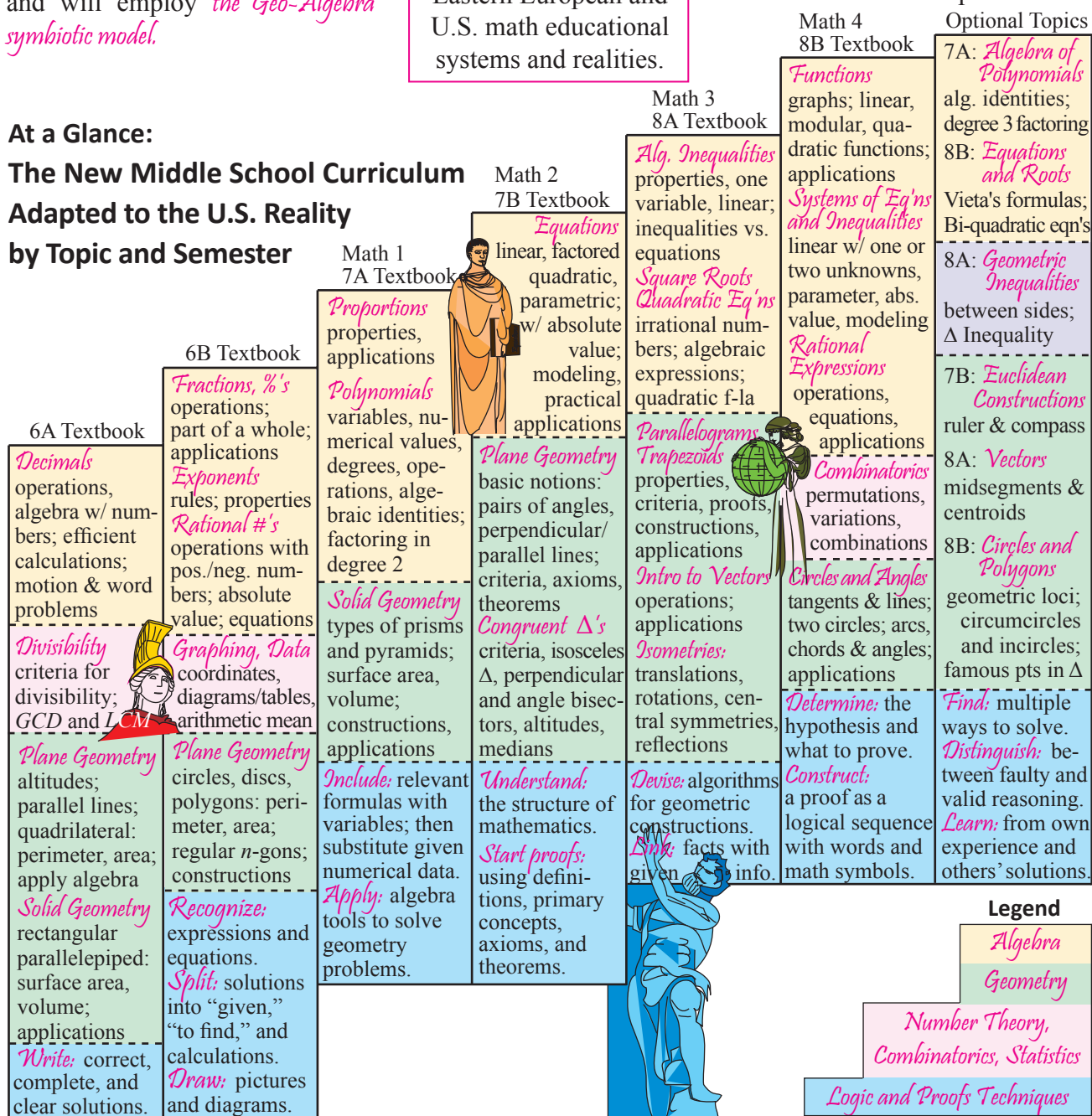


The present textbook series capitalizes on a compromise between Eastern European and U.S. math educational systems and realities.

On the other hand, times **have** changed. *The Bulgarian curriculum has moved part-way towards the U.S. one*; e.g., newer Bulgarian textbooks feature both Algebra and Geometry, while preserving the mathematical integrity and pedagogical insights of a tremendously successful curriculum from the near past.

At a Glance:

The New Middle School Curriculum Adapted to the U.S. Reality by Topic and Semester



8.6. The Main Principles

The **Main Principles** in designing this middle school math curriculum are

Continuity + Balance + Integrity
of the 4 basic building math blocks:



1. **Algebra** is developed from an early stage, using numbers at first and then variables more and more frequently.
2. **Geometry** is developed alongside Algebra as an equal partner.



Algebra & Geometry support and reinforce each other via applications of each subject to the other.



3. **Logic, Proofs, and Problem Solving*** comprise the skeleton of the Math Temple, providing the “glue” that keeps together all parts of mathematics.

In the syllabus on p. XX, the Logic/Proof-Writing Topics are formally split into two semesters in each grade. In reality, they appear uniformly throughout the whole middle school curriculum.

4A. Number Theory, or the Queen of Mathematics,** makes a grand appearance through the topics of *Divisibility*, *Square Roots*, and *Combinatorics* in the 6A, 8A, and 8B textbooks, to aid both logical development and practical algebra skills; e.g., finding *GCD*, *LCM*, number of divisors, etc., efficiently. If you wonder what Number Theory is all about, try to decide if 9,876,543,210 is divisible by 15 *without actually dividing by 15*! The world of *whole* numbers is how Number Theory can be approximately described in middle school.



4B. Combinatorics, Statistics, and Their Applications are ubiquitous in the curriculum, aiming at counting techniques, data graphing, optimization, and approximations, and aided by physical experimentation.

* The role of Logic, Proofs, and Problem Solving is played by the Greek Titan *Atlas*, often pictured “holding the Earth on his shoulders.” In our situation, Atlas represents the *central Pillar of Mathematics*, which keeps the whole structure together.

** Number Theory is widely regarded as the Queen of Mathematics. Indeed, some of the most famous and hard math problems have been in Number Theory; e.g., Fermat’s Last Theorem. In our Math Temple, Number Theory is played by *Athena*, the Greek Goddess of Mathematics (also of wisdom, courage, inspiration, war strategy, arts, literature, and others). In the textbook series, Athena also supports the areas of Graphing and Data Analysis, or Statistics, and other applications.

9. Differentiate and Scaffold

Talking about differentiation in math classes is a “politically correct” and enticing buzzword to mention when advertising a curriculum and a school. Yet, by and large, this talk is not matched by actions to truly support both ends of the spectrum... because it is hard to scaffold math lessons and because *it takes an enthusiastic math specialist to successfully cope with diversity in the math classroom*.

When mathematics is taught:

- in an organized and sound way,
- building a solid Math Temple for everyone,

the number of students who “fall through the cracks” and lag behind is minimized. Students who are ahead, though, need something else: something beyond the regular curriculum that will still keep them topic-wise in the same math vicinity as the rest of their classmates. To this end, the exercises in the **Textbooks** are split into:

- 1] *Basic* (mandatory) problems;
- 2] 3] *High Proficiency* (recommended) problems;
- 4] *Harder/Math Contest* (elective) problems.

The **Workbooks** provide solid “workouts” on the basic material. While everyone should complete a substantial portion of them, the Workbooks are an absolute must for the weaker students, who should work extensively with them.

In addition, a **Problem Collection for the Teacher** is developed for every grade to match the material studied in every chapter of the textbook. Some problems are grouped into multiple-choice tests, and some are open-ended and organized by topic. The material is gradated in three levels:

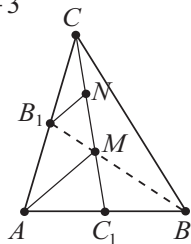
- **Level A:** basic material, building the fundamentals and providing practice for everyone.
- **Level B:** elective material, extending the knowledge of those who want to move forward.
- **Level C:** preparation for math contests for more adventurous and/or advanced students.

9.1. Examples of Scaffolding

The 8B Textbook features the following exercises from the Chapters on *Algebraic/Geometric Inequalities* and *Parallelograms and Trapezoids* that reach all parts of the difficulty spectrum:

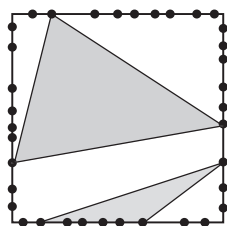
1 (*Basic*). Solve $\frac{x}{x-3} - \frac{54}{x^2-9} = \frac{9}{x+3}$.

- 2** (*High Proficiency*). Points N and M on median CC_1 of $\triangle ABC$ are such that $CN = NM = MC_1$. If B_1 is the midpoint of side AC , prove that B, M , and B_1 lie on one line and $AM \parallel B_1N$.



- 3** (*Hard, but doable with school material*). For which values of the parameter a do the graphs of $f(x) = (a-x)x + (x-2)^2$ and $g(x) = 3(x-2) + 7$ intersect in a point on the x -axis?

- 4** (*More advanced material/ideas*). Given a square w/ 9 points on each side, or a rhombus w/ 12 points on each side, how many $\triangle$'s are formed by these points?



Occasionally, the textbook uses the eye-icon to signify a fact or technique that is *beyond the textbook material*. For example, an **Alternative Proof** in Lesson 155 challenges the reader:

T₂

Theorem (Vieta): If a quadratic equation $ax^2 + bx + c = 0$ has roots x_1 and x_2 , then the sum and the product of the roots satisfy:

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}$$



Alternative Proof: The trinomial factors as $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

Expanding and regrouping the RHS, we obtain:

$$ax^2 + bx + c = ax^2 - a(x_1 + x_2)x + a(x_1 x_2).$$

The two polynomials on the LHS and RHS are equal for *all* values of x . Hence, their corresponding coefficients must also be equal; i.e.,

$$b = -a(x_1 + x_2) \text{ and } c = a(x_1 x_2).$$

Solving for the sum and product of the two roots, we obtain $(x_1 + x_2) = -\frac{b}{a}$ and $x_1 x_2 = \frac{c}{a}$.

9.2. What are the Optional Topics?

This U.S.-adapted program starts in 6th grade with the 5th grade Bulgarian material; that is,

- In the beginning, it lags a *full year* behind its Bulgarian counterpart.
- By the end of 6th grade, it lags only *half a year*.
- By the end of 8th grade, the difference is cut down to only a *third of a year*.

Almost all of the original 5th-8th grade Algebra material and most of the original Geometry material is covered through the current 6th-8th grade textbooks.

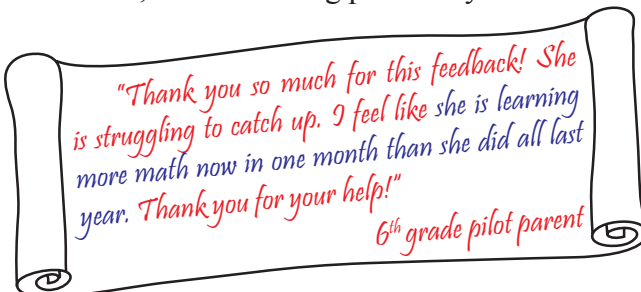
The harder parts of the original 7th-8th grade material on *Degree 3 Polynomials*, *Vieta's Formulas*, *Geometric Inequalities*, *Euclidean Constructions*, *Midsegments and Centroids*, and *Circles and Polygons*, are optional and intended *for the more advanced students* as **elective 7th-8th grade topics**. The optional geometry topics are considerably ahead of the standard U.S. high school material and they will provide:

- *a superb preparation* for those who would like *to be challenged and to excel in Geometry*.

9.3. Does the Scaffolding Work?

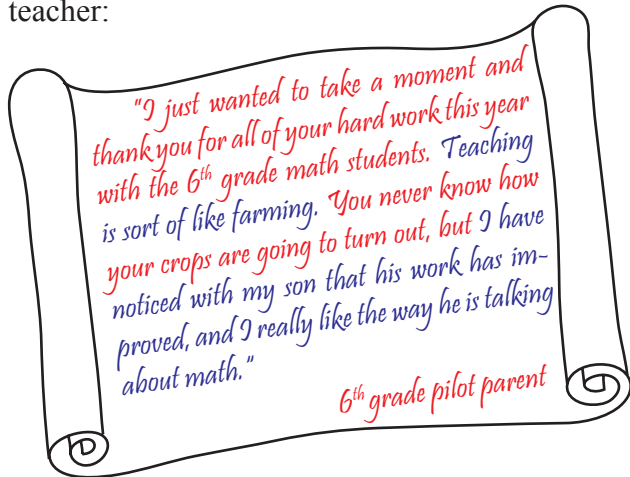
Does the program really benefit all levels of students? Here is some anecdotal evidence.

Answer 1: A month after the start of the pilot 6th grade program, the teacher received a letter from a parent of a student who was struggling tremendously in the beginning, not participating at all in class, and submitting practically blank tests:



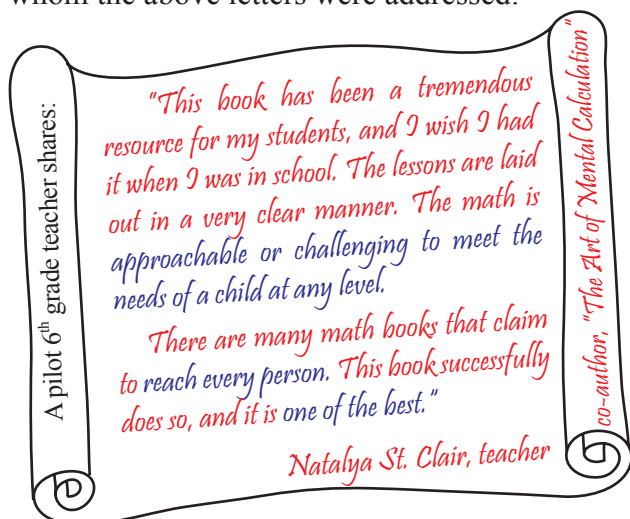
The student gradually progressed to very good performance both on tests and in class, showing enthusiasm and understanding of material to which she had not even been exposed before but was now fired up to learn.

Answer 2: Several months after the start of the pilot program, a parent of a student who had always excelled in math wrote to the 6th grade pilot teacher:



The new program has worked exceptionally well for a number of students who had earlier been identified as having severe problems with learning mathematics. These students have gradually improved and excelled with the new curriculum, and questions about their future success with mathematics have become obsolete. Incidentally, a few of the girls improved so rapidly that 2 months into the program they had to be moved to the more advanced 6th-grade pilot class.

Answer 3: Here is the truth as seen through the eyes of the very first pilot teacher in the U.S. who worked with this program in 6th grade and achieved what was impossible before, and to whom the above letters were addressed:



10. The Value of a Teacher

As with any textbooks, the present textbooks are just that: *a basis for learning*. They will come alive in the classroom *only through the teacher*.

Let us not deceive ourselves: there is no universal panacea for the middle school math problems in the U.S. And although the present textbooks have been hailed by pilot parents as:

- the best textbooks they have ever seen;
- the most suitable for their children;
- a resource to easily follow and learn from at home if a child is absent from school or did not understand something in class...

...still, *a mediocre teacher will do a mediocre job even with the best textbook in hand*.

The fact that many middle school math positions are filled by non-specialists in math is alarming and is one of the major reasons for the current general downfall of U.S. middle school math. A non-specialist does not have an "eagle view" of the full middle and high school math curriculum. Although locally, he/she might be able to teach some of the material well, a lot of damage is done by a teacher's limited knowledge and *lack of understanding of what is important in the long run, even if hard to teach*.

The situation is similar to dropping by parachute the whole math class into the middle of a jungle, giving each student a flashlight, and waiting outside the jungle to see who will come out alive. A better chance of "survival" will be ensured if the students are *given a map of the area, taught how to read and use it, and equipped with the necessary tools, knowledge, and skills* to efficiently cope with the difficulties that will inevitably occur on the way out of the jungle. And *the teacher has to be there, with the students, every step of the way,...* not just waiting for them outside of the "danger zone." I personally never thought of school math in this "jungle vision" until my own children experienced several different U.S. math programs and the image of the "jungle" more and more powerfully resembled the reality of pre-college math education.

More often than not, non-specialist math teachers in middle and elementary school, for a variety of reasons, decide to change the order, depth, and emphasis of the math program they are teaching from. Not knowing the “big picture” of how mathematics will develop over the high school and then college years (and perhaps, not caring about this big picture, since it is not part of their day-to-day job duties), their actions unintentionally destroy the logical structure and goals of the program, whether it is a good program or not. (See earlier discussion of this situation in Section 5: The Temple of Mathematics.)

Fortunately, the new math program:

Is *straightforward to follow*: every lesson is on only 2 pages!

When each lesson in the textbook is on 2 pages only, with 6-to-10 exercises, and when the concepts are practiced in the next lessons and incorporated in new situations for better understanding, it is *very hard to justify staying on one particular lesson for a week or so* (and killing everyone’s enthusiasm in the process).

(Most importantly!)

Ensures accountability from the teachers.

The program is so crystal clear that *anyone can track where the students are and what they have covered* at any time during the school year. By “anyone” we mean parents, administrators, and random visitors alike. It is essential, though, that a copy of *the textbook resides at every student’s home*; there is nothing to hide in the program, while there is so much to learn from reading the textbook at home!

Does not require any search for extra materials elsewhere: the textbook, workbook, and problem collection *contain everything for a successful completion of the program.*

Due to the *natural transparency of the material*, it is impossible to hide behind phrases like “too much,” “too hard,” or “not well organized” “to be covered all in one year.” And the *program does depend on the steady and full completion of the material* in every middle school year, in order for the next year to be successful.

The Mathematical Level

A teacher with a *solid math background* who can *see several steps beyond middle school*, e.g., the high school curriculum, and can recognize:

- the math concepts,
- types of solutions, and
- features of analytical thinking

that are indispensable and must be taught and reinforced in middle school, despite how hard it might be to teach them to a young audience unaware of them.

A Good Choice for a Teacher

for this curriculum would take into account:

The Pedagogical Level

A teacher who can *quickly adapt to new situations* and turn them into an educational advantage for the students.

A teacher who is constantly attuned to the weaknesses and strengths in students’ background and skills, and who accordingly emphasizes features of the curriculum in a balanced way to match and improve the classroom situation.

A teacher who *can communicate mathematics to young students.*

The Level of Dedication

A teacher who is *dedicated to the well-being of the students* and who will sacrifice his/her own desire to “shine on the career ladder” to providing a robust math education to his/her young charges.

A teacher *who believes in and is ready to undertake and follow the new curriculum closely*; who is ready to face the mathematical and cultural difficulties that will inevitably arise in introducing such a program.

A teacher who *will persevere.*

11. Can a U.S. Middle or High School be Successful with this New Math Curriculum?

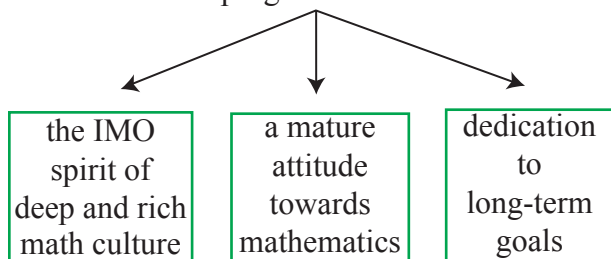
What kind of question is this?! Let us analyze the situation as a mathematician would, putting aside our own ambitions and de-politicizing math education.

11.1. The U.S. Gene Pool

Of course, on the average, U.S. students can do just as well as (if not better than) their Bulgarian peers, and, for that matter, any peers around the world. As far as the “gene pool” of the U.S. is concerned, it is probably the best mixture in the world.

The cream of the crop among U.S. students have performed at the top level and have occasionally made the number 1 team in the world (!) at the International Mathematical Olympiads (IMO)*, the preeminent world math competition for pre-college students that tests **not** speed but *depth and originality of mathematical thinking*.

The new math program emulates:



It is true that very few students will end up being math Olympians, and we will not give false hopes that the current program is training students for the IMOs. However, the expectation is that:

Everyone who is touched by this program will build a solid Temple of Mathematics, where later, if desired, we may hang “fancy chandeliers” and “brilliant art pieces.” More importantly, the personal connection with mathematics will remain all through life to be enjoyed and cherished.

11.2. Can Girls Succeed in Math?

Upon coming to the U.S., I was shocked by this question. I was raised in a provincial town in Bulgaria and made it as a high school student to two IMOs, winning silver medals. There was another girl on the Bulgarian team for the 2 years I was there.

It took the U.S. 25 years of participating at the IMOs before the first U.S. girl, Melanie Wood, qualified for the IMOs. I was privileged to train the U.S. national team when Melanie competed at the IMOs, also earning two silver medals. Later, I participated in the training of the other two U.S. girls who went on to win gold medals at the IMOs.

To cut the long story short: anything is possible and *gender does not matter as far as mathematical talent and success are concerned*.



Gender apparently matters in the social and cultural aspects of education, and this severely handicaps many U.S. girls who might otherwise have become excellent mathematicians.



Decades after competing at the IMOs, I learned that Bulgaria is the top country in the world in sending a total of 21 girls so far** to the IMOs. Germany and Russia are next, with 19 and 15 girls, respectively. The USA has only 3 girls so far.

Perhaps *it is time for the U.S. educational system to follow the example of other programs from around the world* that have been hugely successful in raising gen-

* The IMO takes place every summer in a different country. Each country’s team has 6 students. As opposed to standardized tests, there are only 6 problems over 2 days, 4.5 hours each day. The solutions are written as rigorous mathematical proofs.

** See “Cross-Cultural Analysis of Students with Exceptional Talent in Mathematical Problem Solving” by Titu Andreescu, Joseph A. Gallian, and Jonathan M. Kane, <http://www.ams.org/notices/200810/fea-gallian.pdf>.

erations of women who are highly educated in math and other STEM (Science, Technology, Engineering and Mathematics) disciplines, and who have been brought up to think of men and women as intellectually equal.

11.3. The Needed Cultural Shift

The current middle school math program originating in Bulgaria necessitates a cultural shift in the U.S. school community so that the program can take root and yield results. For starters, the school must make:

- *math a top priority in its academic curriculum*

and continue unwaveringly in its determination to overturn the middle school math situation to the huge benefit of its students.

11.4. Continuity of the Curriculum

It is most important and advantageous for everyone that:

The **same** teacher continues with the **same** students throughout middle school.

The alternative has struck me as a very bad way to organize middle school math education. Indeed, when a person teaches the same grade year after year he/she:

- gains no understanding of the “big picture” of pre-college math education;
- shares no responsibility for the continuation of the program in the next grade (and a lot of time will be wasted in the beginning of each school year for the new teacher to test and get attuned to the new students);
- has no incentive (or responsibility) for reaching the final goals of the program.

Yes, I am acutely aware of the counterargument:

“I do not want a bad math teacher to teach my kid for all of middle school. I would rather bite the bullet of a bad math teacher one year, so as to get the good math teacher the other year.”

In summary, many parents are (unwittingly) willing for their kids to endure on the average a mediocre middle school education. Can you imagine what kind of damage the “bad” teacher would do to the Temple of Mathematics during that, hopefully, single year of “bad” math lessons? Would the next, “good” teacher have the opportunity, or even the incentive, to rebuild the Temple AND cover the new material on top of that? Probably not.

This kind of “averaging math teaching” is another major reason for the tremendous difficulties in U.S. middle school math education. *A teacher must be an asset for math education*; equally important (if not more important) than the actual math curriculum! And so, the truth must be spoken:

- a bad math teacher must be let go, to prevent more irreversible damage to our kids;
- a good math teacher must be treasured and given opportunities to further develop.

11.5. Local Support

There is no easy remedy and no miracle solution to math education problems anywhere around the world. The new math Program will need all the support it can get from parents and students, in addition to teachers and administrators. The Program will depend on the school community’s belief in it and commitment to it.

Approaching the end of the first pilot year, here is what the Head of School shared, after extensive observations and conversations with pilot teachers, parents, and students:

“I have only heard GREAT comments about math during and after parent-teacher conferences in March. People are very pleased with the direction things are going with the Bulgarian curriculum. We are very excited to continue to grow the program next year!”

*Debra Massey
Former Head of Pilot School
Tehiyah Day School, El Cerrito, CA*

12. Within the Global U.S. Picture

At a sectional meeting of the Mathematical Association of America (MAA) in 2014, I heard a captivating presentation by *Dr. Phil Daro*, one of the three principal writers of *the Common Core Mathematics Standards* (CCSS-M). The talk addressed the biggest education problem in school mathematics, on which the CCSS-M writing team focused:

"The mile wide, inch deep" configuration of the American school curriculum.

As a consequence, Dr. Daro would ask any teacher:

"Are you teaching

Mathematics or *Answer-Getting Strategies?*"

This question fully resonated with the educational philosophy with which I was brought up back in Bulgaria. Dr. Daro explained that the core change incorporated in the CCSS-M is:

A shift in the perspective toward mathematics taught in schools:

Less emphasis on a fragmented multitude of special solution methods, especially tricks that avoid underlying mathematical principles.

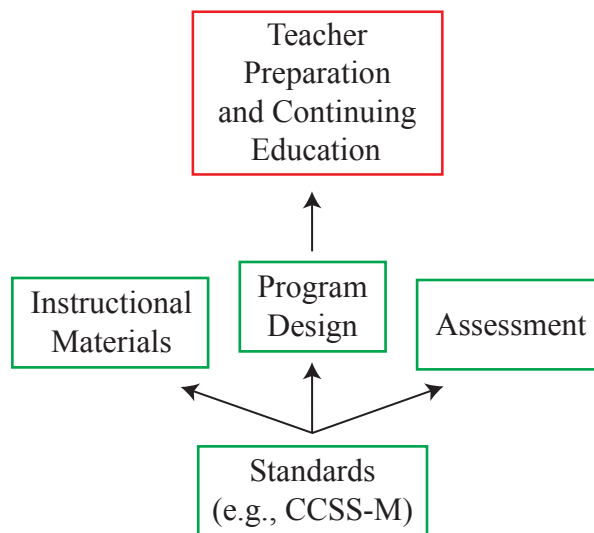
More emphasis on core principles and practices that unify and focus mathematical thinking.

Dr. Daro assembled the major pieces of the gigantic math education picture in the U.S.:

"Standards can help solve this kind of curriculum architecture problem, but we all face more detailed and difficult challenges of instructional materials, assessment, and program design, as well as the even more [difficult] area of teacher preparation and continuing education."

*Dr. Phil Daro
A Principal Writer
Common Core Mathematics Standards*

Visually, the scheme below represents the levels of change in the U.S. math education that are envisioned by the CCSS-M writing team:



At the time, I had no idea that I would soon launch the translation, adaptation, and implementation effort of the present curriculum. Inspired by Dr. Daro's talk, I invited him to meet the then-leaders of Tehiyah Day School in person. Perhaps, not entirely by serendipity, this meeting marked the path for the new math curriculum: as it turned out, the new Bulgarian curriculum matched the CCSS-M very closely.

During the 2016-2017 academic school year, I spent a day every week working with the pilot classes, training the pilot teachers, and learning myself how to adapt the curriculum to the U.S. reality. The instructional materials, program design, and assessment of the adapted math curriculum for 6th-7th grades have been published, and the adaptation of the 8th grade curriculum is ready.

Coming full circle, two years after his memorable MAA presentation, Dr. Daro reviewed the U.S.-adapted version of our new math program and assessed it as a

"Middle and High School Math Program that is deeply aligned with the Common Core Mathematics Standards".

13. Math 4 (8B) Program by Topics and Lessons



* L87-88 = Lessons 87 through 88

Concepts and Logic

Basic a Logic Structure of Mathematics:

- Points vs. Pairs of Coordinates in the Plane L83
- Conjecture vs. Mathematical Proof L88
- Impossibilities: Vertical Line Test L85,89
- Domains: Implications for Functions L86-87,92
- Change in Definitions and Consequences: L184
- To Solve an Equation or Inequality: to Find the Solutions or Determine There are None L152
- Alternative Proofs and Converse Theorems L155
- Unique, Exactly One, No Object L85,104,136

Build a Logic Network in Circle Geometry:

- Circles, Secants, and Tangents L121-123
- Circles, Chords, and Arcs L124-125
- Central, Inscribed, Peripheral Angles L124-129
- Circumcircles and Incircles of Triangles and Quadrilaterals L136-139,143-146
- Famous Points in Triangles L140-141

Build a Logic Network in Geometric Constructions:

- Elementary and Basic Constructions L132
- Solving Construction Problems L132-133
 - Analysis, Algorithm, Proof, Discussion
 - Position-Free vs. Positional Problem
 - Same solution “Up To Congruence”
- Geometric Locus of Points L134-135,142
 - GLP that Oversee a Segment at an Angle
 - Constructing a Triangle

Problem Solving Techniques

Use Finitely Many Points to Draw Conclusions:

- In Tables for Functions L84
- In Making Predictions about Next Points L85

Multiple Solutions in:

- Construction Problems L133,135
- Geometric Proofs L143
- Problems with Equations/Functions L105,156

Reduce to Previous Problems/Situations:

- Repeated Expressions L152-155
- in Circle Geometry L123,139-141,143,146-147
- in Algebra L151,161-162,166

Create and Write Proofs

- “If and Only If” L121,125,144-146
- By Contradiction L85,126
- Extra Construction L129,141,148
- Identifying Combinatorial Type L176
- Algorithms L105-106,129,132-135,142,161

Build a Logic Network in Functions:

- Inputs, Values, Domains, Graphs L84-85
- Types/Graphs: Direct/Inverse Variations, Linear, Parabolas, Hyperbolas, Modular L84-97
- Connect Functions, Equations, and Inequalities through the Graphing and Geometry L94-95

Build a Logic Network in Systems:

- Systems of Two Equations w/ Two Unknowns:
 - Linear Equations and Their Graphs L102
 - Equivalent Systems L102
- Systems of Two Inequalities w/ One Unknown:
 - Unions/Intersections of Intervals L113-117
 - Product and Modular Inequalities L115-116

Build a Logic Network in Equations and Roots:

- Quadratic Formula and a Shortcut L151
- Factoring Trinomials L151
- Vieta's Formula and a Shortcut L155-156
- Equation Reducing to Quadratics L152-154

Build a Logic Network in Combinatorics:

- Operations with Possibilities L171-172
- Permutations L173
- Variations L174
- Combinations L175

Split into Cases when Solving and/or Proving:

- Signs of Roots of Equations L156
- Circle Geometry L122,126,128,134-136,143
- Rational Expressions L162,166-167
- Combinatorics and Number Theory L172

Organize Data/Knowledge by Making:

- a Table L84-99,102-104,107-108,139,156,168, L178,180,181,184,187
- a Graph L84-99,102-104,107,178,182-184
- a List L132,140,157,159,179,187
- a Diagram L84,98-99,106,108,112-118, L171-172,176,179,181-185,187,190
- a Collection of Pictures L98-99,109,112,118, L121-148,178,182-183,185-189

Study and Review

Review: L83,132,178-190

Summarize: L98-99,109,130,147,157,168

Chapter Tests: L100,110,131,150,158,169,177

Problems: L101,111,120,149,170

Exit Tests: L191



14. How to Work w/ Math 3 (8B) Materials

The **Math 4/8B Textbook** is for use in class as the main source of material to be covered:

- The *Textbook Examples* are the core of the class discussion and work.
- The *Textbook Exercises* can be done in class or assigned for homework.
- *Answers to all Exercises* are in the back of the textbook; hence, writing only answers will bring no credit, while good reasoning and organized calculations will earn a lot of credit.

All *Lesson Exercises* in the **Math 4/8B Workbook** (marked as L__) are identical to the examples in the textbook, with spaces for writing solutions. The Lesson Exercises are intended for classwork.

The *Worksheets* in the **Math 4/8B Workbook** (marked as W__) thematically match the textbook, lesson-by-lesson. The Worksheets are intended mostly for independent work in class or homework. They are usually (but not always!) geared towards students who need more help and extra support with the material.

The **Math 4/8B Problem Collection for the Teacher** can be used for tests and supplementary work with students who excel with the material.

The **Math 4/8B Pedagogical Guide** will reveal to the teacher ideas and strategies gained by decades of teaching in the U.S. and by the practical wisdom and experience of the Program's pilot teachers.

15. Biographical Sketches

The idea for the *Archimedes Publishing House* was conceived in 1982 when Professor Georgi Paskalev collected all entrance exams given thus far to Mathematics Departments in universities across Bulgaria and wrote a book with their solutions. The occasion was the upcoming exam of his own son, who actively participated in the preparation of the book. Until 2004, Professor Georgi Paskalev continued to study the competitive university entrance exams and published a variety of preparation materials. These books have enjoyed continual interest, large circulations, and many editions.

In 1989 Professor Georgi Paskalev, together with his wife Zdravka Paskaleva and the grown-up children Plamen and Nadezhda, founded publishing house "*Georgi Paskalev*," specializing entirely in mathematics materials. Ten years later Plamen Paskalev assembled a team of authors and published a collection of tests to prepare for admissions to high schools. This book became the first publication of "*Archimedes*," the successor to "*Georgi Paskalev*." The new publishing house is directed by Plamen Paskalev, and it still bases its activities on the creative duo of Georgi Paskalev and Zdravka Paskaleva. During 1997-2004, these two authors created a complete series of math textbooks for 5th-12th grades, which set new standards in the methodology of teaching mathematics and has ever since enjoyed popularity in teaching circles around Bulgaria.

Over the years, Archimedes Publishing House grew to include books in informatics, information technology, and chemistry, as well as translated academic and journalistic literature. "Archimedes" is an active member of *Association "Bulgarian Book"* and an active participant in the publishing conferences of the *Union of Mathematicians in Bulgaria*. In 2010 the *European Council for Scientific and Cultural Community* awarded the "*Golden Book*" to "Archimedes" for its contributions to the development of Bulgarian culture.

Professor Georgi Paskalev (1932-2004) taught for many years at Sofia University. He co-founded the Mathematics Department at Plovdiv University (PU) and was the Vice Provost of PU (1970-1972) and also a longtime Chair of the Departments of Mathematics and Mechanics at PU and the University of Structural Engineering & Architecture (USEA) in Sofia.

Professor Paskalev was known for his ability to teach difficult math disciplines in a clear and accessible manner. He published numerous research papers in various areas of mathematics and mechanics. He mentored many mathematicians and authored more than 100 books, problems collections, and textbooks in mathematics. In 2004 the Academic Council of PU named its auditorium after him. Another auditorium at USEA in Sofia is also named in his honor.

Edravka Paskaleva (1938-2017) was a well-known and experienced specialist in mathematics and the methodology of mathematics, with 20 years experience as a teacher in leading high schools and 10 years as a leader in teachers' education and development, having observed and reviewed over 1,000 lessons by teachers at various schools. From 1985 until 2010 she was the head of the Education Center "Archimedes".

Maya Alashka received a degree in mathematics from Sofia University (SU), with a specialty in elementary school education. She has taught math for 40 years at all pre-college levels. As an affiliate of SU, she has trained many future teachers. She has participated in various education projects and co-authored a number of popular textbooks, problem collections, and other education materials for all pre-college levels.

Plamen Paskalev is an Assistant Professor of Mathematics, author of dozens of scientific publications in mathematics and mechanics, collections and guides for students, and articles on the challenges in math education. He has taught at USEA, the University of National and World Economy, and the German Language High School in Sofia. He has directed "Archimedes" since its creation in 1999. In 2004 he founded and has since chaired the Union of Publishers of Textbooks & Curriculum Materials in Bulgaria.

Professor Zvezdelina Stankova (Zvezda) was drawn into the world of mathematics when, as a 5th grader, she joined the math circle at her school in Bulgaria and, three months later, won the Regional Math Olympiad. She represented her home country at two International Mathematical Olympiads (IMOs), earning silver medals.

As a freshman at Sofia University, Zvezda won a competition to study in the U.S. and completed her undergraduate degree at Bryn Mawr College in 1993. She did her first math research in enumerative combinatorics at two summer programs in Duluth, Minnesota. The resulting papers contributed to her *Alice T. Schafer Prize* for Excellence in Mathematics by an Undergraduate Woman, awarded by the Association for Women in Mathematics. In 1997, Zvezda received a Ph.D. from Harvard University, with a thesis on moduli spaces of curves, in the field of algebraic geometry. She also earned a high school teaching certification in the state of Massachusetts and later in California.

As a postdoctoral fellow at the Mathematical Sciences Research Institute (MSRI) and UC Berkeley in 1997-1999, Zvezda co-founded the Bay Area Mathematical Olympiad and created the Berkeley Math Circle (BMC). She trained the USA national team for the IMOs for six years, including the memorable year of 2001 when three of the six team members were BMCers and the USA tied with Russia for second place in the world overall. She worked at Mills College 1999-2016, and she joined the faculty at UC Berkeley in the summer of 2016.

Zvezda's inspiring style and passion to teach were recognized by the Mathematical Association of America (MAA): in 2004 she received the first *Henry L. Alder Award* for Distinguished Teaching by a Beginning College or University Mathematics Faculty Member. In 2011 MAA honored her with the highest math teaching award in the U.S., the *Deborah and Franklin Tepper Haimo Award* for Distinguished College or University Teachers who are widely recognized as extraordinarily successful and whose teaching effectiveness has shown to have influence beyond their own institutions. Zvezda was featured in the *Salutes Program of the ABC 7 News* in spring 2011. In 2012, she was listed in *Princeton Review's* "300 Best Professors."

In fall 2015 she introduced a new middle school math program at Tehiyah Day School in El Cerrito, CA, and in fall 2016 she founded the "Math Taught the Right Way" program for middle and high school students at UC Berkeley. Both programs are based on this textbook series.

Zvezda's most enduring passion remains working at the BMC with young students motivated to discover new mathematical wonders. She spends a lot of time with her girl and boy, studying foreign languages and playing the piano with them, and teaching them mathematics in the "Bulgarian" way.

Tom Rike graduated from San Francisco State College in 1968. He taught in the Oakland Unified School District from 1970 until he retired in 2003 and now volunteers at Oakland High School three days a week. He served as the high school liaison for BMC and BAMO until 2009.

Tom has been fascinated by giants of mathematical thought such as *Archimedes*, *Euler*, and *Gauss*, and has shared their works at the BMC sessions on a number of occasions: using the *arbelos* from the *Book of Lemmas*

by Archimedes; showing Euler's solution to the *Basel Problem*; and demonstrating Gauss's proof from *Disquisitiones Arithmeticae* that a 17-gon is *constructible*. It is Archimedes' *lever*, on which he "balanced" his *Mass Point* article in Volume I of *A Decade of the Berkeley Math Circle - the American Experience*. For a number of years, Tom also ran a math circle at Oakland High.

Tom has been involved as a coach and director of the *East Bay Mathletes*, a monthly competition among local high schools since 1978. In 1983, he helped found in Oakland a middle school math competition, *All-Star Mathletes*. Among his other interests are the game of go, opera, chamber music, languages, and the writings of *James Joyce*.

The current series of textbooks has tremendously benefited from Tom's unsurpassed expertise in U.S. pre-college mathematics and from his generous and unbounded enthusiasm for new and sound ways to teach mathematics to youngsters.

Kelli Talaska has been a student and teacher of mathematics nearly all her life, starting with frequently volunteering to write the problem of the day in fifth grade. Her mathematics education really took off in her years as a student and later a counselor at the *Honors Summer Math Camp* (for high school students) at Texas State University. She spent two summers working on original mathematics research at the *Duluth Summer Program* for undergraduates. After completing her degree at Texas A&M University in 2002, Kelli spent two years teaching at Lowell High School in San Francisco.

She later returned to graduate school at the University of Michigan and received her Ph.D. in 2010, with a thesis in algebraic combinatorics. Kelli was an NSF Postdoctoral Fellow at UC Berkeley from 2010-2013, and returned to UC Berkeley as a lecturer in 2015, after starting her tutoring and enrichment business, *Octopus Math*.

Kelli relishes the chance to work with students of all ages and to expose students to the idea that mathematics involves not just algorithms and memorization, but also incredibly rich opportunities to experiment and to look for patterns and connections. She has been involved with the *Berkeley Math Circle* since 2015 and has also had the opportunity to teach at the *Santa Cruz Math Circle* and at *Epsilon Camp*.

Stankova's and Rike's bios of are partially taken from "A Decade of The Berkeley Math Circle," AMS/ MSRI's Math Circles Library.

Kelli really enjoys hands-on math explorations, including logic puzzles, geometric constructions, and paper cutting and folding activities. Outside of math, Kelli is an avid scuba diver and loves to talk about fascinating ocean critters.

Norm Prokup works as a math teacher in College Preparatory School in Oakland, CA. His short bio is included on page 4 of this Introduction Chapter, along with his quote regarding the translated and adapted Bulgarian Math Program.

Elysée Wilson-Egolf, although always fond of and interested in mathematics, spent most of her early schooling years studying literature and world languages. Strong encouragement by her grandfathers and her parents' provision of practical education in woodworking, repair, and agriculture led her to apply for study in college as a mechanical engineer. As an undergraduate at *UC Berkeley* she was exposed to real mathematics for the first time. She immediately applied for a change in degree to mathematics major and was soon spending full days in the math/stats library in Evans Hall in UCB campus.

After graduating with her BA in 2013, she remembered the patience and skill of her Russian and Polish instructors with gratitude and looked for opportunities to share their mathematical taste with others. She built on her years of after-school tutoring as a teenager and started working with the Beginners' group at the *Berkeley Math Circle's Summer Program* in 2016. Subsequently she has taught 6th grade algebra and both 6th grade and 7th grade problem-solving with MTRW; at BMC, she has taught Beginners I and II and Intermediate I groups. Although she likes to share any kind of elegant proof or surprising result, she is especially fond of classical geometry, topology, and analysis: her favorite lessons with BMC involve repeating decimal expansions and limits, the combinatorics of 4th-dimensional objects, and graph theory. A quote by Elysée regarding the translated and adapted Bulgarian Math Program appears earlier in this chapter.

Since the fall of 2016 she has also worked with *Bentley Upper School*, where she uses the Bulgarian textbooks to teach Honors Geometry and enjoys tying school topics to their applications in practical living.

When not learning more mathematics or teaching her students, she still translates poems and walks in the mountains.

16. Math Taught the Right Way

In addition to running the Tehiyah Day School math program on this textbook series for 3 years, we continued the weekly program “*Math Taught the Right Way*” (MTRW) for a 3rd year, extending the Berkeley Math Circle at UC Berkeley.

In fall 2018, MTRW had an increased enrollment of 135 students from 90 schools and 20 towns in the extended San Francisco Bay Area. The four program groups were selected mainly by age. As in the spirit of the Berkeley Math Circle, younger, more advanced students were promoted to higher groups. The fall groups were named 6A, 7A, 8A, and 9A.

6A Group:

- 6A Algebra
- 6A Geometry
- 6A Problem Solving

Taught by:

Elena Blanter
Harry Main-Luu
Ellen Kulinsky

7A Group:

- 7A Algebra
- 7A Geometry
- 7A Problem Solving

Elena Blanter
Harry Main-Luu
Elysée W.-Egolf

8A Group:

- 8A Algebra
- 8A Geometry
- 8A Problem Solving

Kelli Talaska
Norm Prokup
Elysée W.-Egolf

9A Group:

- 9A Algebra
- 9A Geometry
- 9A Proofs

Kelli Talaska
Norm Prokup
Zvezdelina Stankova

In the 15 weekly 3-hour sessions in fall 2018,

- The 6A group covered almost all of the 6A curriculum. 6A Problem Solving concentrated on the *Divisibility* chapter and *Word Problems* from the 6A curriculum.
- 7A Algebra covered *Proportions* and *Integer Expressions*. 7A Geometry worked on *Polyhedral* and *Round Solids* in the 7A curriculum, and *Foundations of Geometry* in the 7B curriculum. 7A Problem Solving worked its way through *Bulgarian Math Contests* for 6th-7th grades, learning problem-solving strategies and practicing ideas like solving through graphing and algebra, applying geometry theorems and formulas for difference and sum of squares formulas, and multiplying polynomials.

- 8A Algebra covered *Equations Part II* (*Modeling and Applications*) from the 7B curriculum, and *Algebraic Inequalities* from the 8A curriculum. 8A Geometry covered *Congruent Triangles, Part I* and *Parallelograms and Trapezoids* from the 8A curriculum. 8A Problem-Solving was based on *Combinatorics* from the 8B curriculum and from *A Decade of the Berkeley Math Circle - the American Experience*, vol. I, edited by me and Tom Rike, and generalized to *Combinatorics of an n-dimensional cube analogue*.
- 9A Algebra covered *Combinatorics* and *Equations and Roots* from the 8B curriculum, *Classical Probabilities* from the 9A curriculum, and practiced with *Rational Equations*. 9A Geometry covered 2/3 of *Circle Geometry Parts I-II* from the 8B curriculum. 9A Proofs covered *Rational Inequalities* from the 9A curriculum and *Inequalities, Part I* on AM-GM and other power mean inequalities, and the beginning of *Smoothing Inequalities* from “*A Decade of the Berkeley Math Circle - the American Experience*,” vol. I-II, edited by me and Tom Rike.

The Berkeley Math Circle Summer Program 2018 can be viewed as a third component of the MTRW Program, which completes the 6th-9th grade curriculum, as well as some math circle topics. The faculty who taught at the Summer Program were:

- Chris Overton, Ph.D., Ayasdi
- Ellen Kulinsky, UC Berkeley
- Elysée W.-Egolf, B.A. UCB, Bentley High
- Harry Main-Luu, BMC asst, UC Berkeley
- Joshua Zucker, M.S. Stanford
- Kelli Talaska, Ph.D., UC Berkeley
- Laura Pierson, BMC alumna, Harvard
- Maia Averett, Ph.D., Mills College
- Matyas Sustik, Ph.D., WalmartLabs
- Norm Prokup, Ph.D., College Prep, Oakland
- Quan Lam, Ph.D., MBA, UC President’s Office
- Zvezdelina Stankova, Ph.D., UC Berkeley

17. Acknowledgments

The adaptation of language and math notation for the U.S. reality in this middle and high school math program, as well as the technical and mathematical correctness of the series, is due to the dedication and selflessness of *Tom Rike*, my long-time collaborator and co-editor, and *Kelli Talaska*, *Norm Prokup*, and *Elysée Wilson-Egolf*, instructors at BMC and MTRW. You are all *amazing*!

Natalya St. Clair was the very first pilot teacher, who moved across the country to bravely undertake the new math Program in fall 2015 with the two 6th grade classes at Tehiyah Day School. Her enthusiasm for mathematics was evident each and every day in her lively lessons, sharing with her students her joy in teaching math. She also had numerous suggestions for adaptation based on her daily experience with the Program on U.S. turf, and offered constructive criticism of the revised version of the series. *George Khasin* and *Laurie Gold* continued the pilot Program in spring 2016 with a steady rhythm and care for the students, and *Andrea Jedwab* piloted the 7th and 8th grade and continued with the 6th and 7th grade classes in 2016-2018.

Going further back, *Elise Prowse*, the former Interim Head of the pilot school and a math educator herself, recognized the power and depth of the new Program (while it was still in the original Bulgarian language!) and laid down the path of its implementation in practice. *Debra Massey*, the former Head of School and a parent herself, strongly backed the Program and supported it all the way through the crucial first two pilot years.

The middle school math Program at Tehiyah Day School would not have had a stable basis to land on had it not been for the Singapore Math Program in grades K-5, introduced in 2003 by *Professor Alexander Givental* of UC Berkeley and his wife, *Laura Givental*, the leader of the Berkeley Math Circle-Elementary Division for grades 1-4. The mere fact that Singapore Math preceded the pilot Math Program already immensely increased the chances of success of this pioneering Project.

It goes without saying that the vision of *Professor Georgi Paskalev* and his collaborators, *Zdravka Paskaleva* and *Maya Alashka*, has provided the generating power for this translation and adaptation effort. The success of the original Bulgarian textbook series is no surprise. The novel ideas of the overall design, the simplicity and elegance in its organization, and the balance between depth and breath of the material, are unsurpassed.

Any textbook series needs a Publisher. And what more suitable Publisher of the new textbook series is there than the original “Archimedes” Publishing House! Its Director, *Professor Plamen Paskalev*, undertook the Project without reservations, investing in a future based on the common ground of two very different math educational systems: the U.S. and the Bulgarian systems.

Angelina Avramova, the graphic designer of the original series and the technical editor of the new series, prepared the materials for publication with efficiency and professionalism that would put any anxious collaborator at ease and that would make any publisher proud.

Looking forward, *Dr. Phil Daro*, one of the three Principal Writers of the Common Core Standards, reviewed and endorsed the translated and adapted Program. *Dr. Doug Ensley*, the Deputy Executive Director of the American Math Competitions made available about a dozen AMC8 problems for inclusion in the current 7B Textbook.

The daily work of the pilot *6th-8th grade students* and *the constant support of their parents* at Tehiyah moved this program from paper to real life and gave it meaning and purpose. The translation and adaptation effort would have been inconceivable without the understanding and love of my family: my husband, *Dima*, and our children, *Lily* and *Oleg*.

With gratitude,
Zvezdelina Stankova
Translator, Adapter, and Co-Author

85. Graph of a Function

Example 1 The distance from Sofia to Plovdiv (two major cities in Bulgaria) is 140 km. A car starts from Sofia to Plovdiv and moves at a uniform speed of 70 km/h.

- Find the distance that the car travels in t min.
- Find the distance that the car travels in t min, if in the middle of the trip it stops for half an hour and then continues with the same speed.

Solution:

$$\text{a) } v = 70 \text{ km/h} = \frac{70 \text{ km}}{1 \text{ h}} = \frac{70 \text{ km}}{60 \text{ min}} = \frac{7}{6} \text{ km/min}$$

Then by the formula $s = vt$ we find that the distance covered in kilometers after t min is $s = \frac{7}{6}t$.

The car covers the distance from Sofia to Plovdiv for $t = \frac{140}{70} = 2$ h, or 120 min. Thus, the time t can take on values in minutes from 0 to 120; i.e., $t \in [0, 120]$.

- The road from Sofia to Plovdiv is 140 km and is covered in 120 min. Therefore, half of the road is 70 km and it is covered in 60 min.

In the time interval $t \in [0, 60]$ the distance covered by the car in t min will be found by the rule $s = \frac{7}{6}t$.

In the time interval $t \in [60, 90]$ the car does not move and the distance covered is still 70 km; i.e., $s = 70$ km.

The stopover prolongs the overall time by 30 min and the car covers the distance from Sofia to Plovdiv for a total of 150 min.

In the time interval $t \in [90, 150]$ the road covered in kilometers will be $70 + (t - 90)\frac{7}{6}$; i.e., $s = \frac{7}{6}t - 35$.

The rule for finding s by a given time t is:

$$s = \begin{cases} \frac{7}{6}t & \text{when } 0 \leq t \leq 60, \\ 70 & \text{when } 60 < t \leq 90, \\ \frac{7}{6}t - 35 & \text{when } 90 < t \leq 150. \end{cases}$$

In *Example 1* the distance that the car covers in t min when going from Sofia to Plovdiv was found for two different versions of the trip.



In part b), to find the distance of the car from Sofia at the chosen moment t , we use different expressions depending on the interval to which t belongs.

For example, to find the distance from Sofia traveled by the car:

- in 30 min, we set $t = 30 \in [0, 60]$ and use the expression $\frac{7}{6}t$ to find $s = \frac{7}{6} \cdot 30 = 35$ km;
- in 80 min, we set $t = 80 \in [60, 90]$; the car will be 70 km from Sofia because it has stopped moving;
- in 120 min, we set $t = 120 \in [90, 150]$ and use the expression $\left(\frac{7}{6}t - 35\right)$ to find that the car is at 105 km from Sofia. $\left(s = \frac{7}{6} \cdot 120 - 35 = 105\right)$

In part (a) for an arbitrary value of $t \in [0, 120]$ we found s by the rule $s = \frac{7}{6}t$. In the first version of the trip, the function $y = f(x)$ is defined with domain $D: x \in [0, 120]$ and by the rule $f(x) = \frac{7}{6}x$.

For every value of $x \in D$ we obtained exactly one value of y :

x	0	6	12	60	90	120	$x \in D$, arbitrarily chosen value
$y = f(x)$	0	7	14	70	105	140	

If we choose a rectangular coordinate system in the plane, the ordered pairs of numbers $(x, y = f(x))$ are represented in this plane by the points $M_1(0, 0)$, $M_2(6, 7)$, $M_3(12, 14)$, ..., $M_6(120, 140)$. (Fig. 8)



It is interesting to note that the six points we choose are positioned on the same line. Naturally, we expect that if we construct other points $M(x, f(x))$ using the same function, then they will also be on the same line; more precisely, in our present situation, they would lie on segment M_1M_6 since the domain is $D: x \in [0, 120]$.

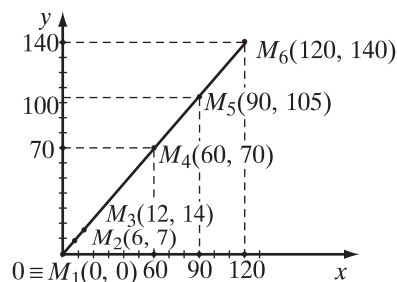


Fig. 8

D8 Given the function $y = f(x)$, $x \in D$, and a chosen rectangular coordinate system Oxy in the plane, for all $x \in D$ we calculate the corresponding value of $y = f(x)$ and we mark the point M with coordinates x and y ; i.e., $M(x, f(x))$. The set of all such points M is called **the graph of the function $y = f(x)$, $x \in D$** .

In part (b) in *Example 1* the graph of the function $S(y)$ with argument $t(x)$ consists of three segments OA , AB , and BC (Fig. 9). To draw the graph of this function, we find the ordered pairs (x, y) that are the coordinates of the points on the graph:

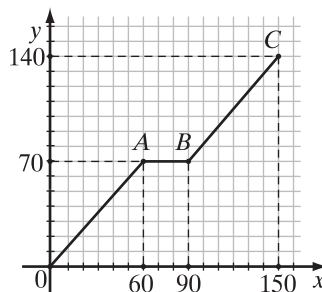
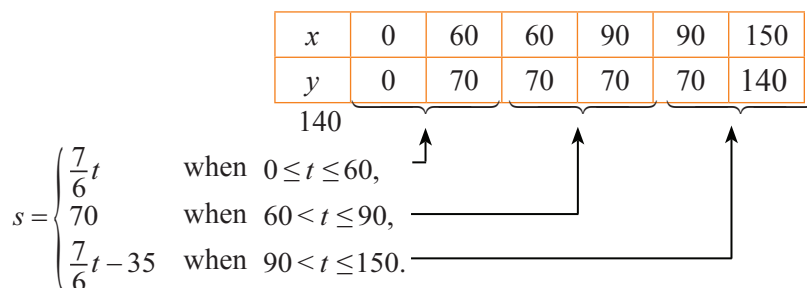


Fig. 9

Example 2

During the school year 1957-1958, the grading system in Bulgaria changed from the set of grades $\{2, 3, 4, 5\}$ to the set of grades $\{2, 3, 4, 5, 6\}$. The re-scaling of grades was done by the formula $\frac{3}{2}x - \frac{3}{2}$, where x took on the values 3, 4, and 5. The old grade of 2 remained the same. Represent the grades other than 2 in the new system as a function of the grades in the old system and draw its graph.

Solution:

The function is $y = \frac{3}{2}x - \frac{3}{2}$ with $D: x \in \{3, 4, 5\}$. The domain of the function consists of three values for x . So the function also has only three values and its graph consists of three points A , B , and C . (Fig. 10)

$x \in D$	3	4	5
$y = \frac{3}{2}x - \frac{3}{2}$	3	4.5	6

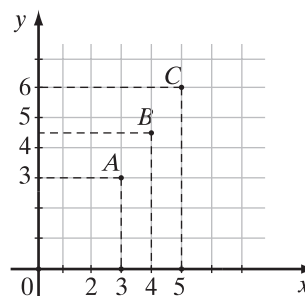


Fig. 10



In Figures 8, 9, and 10 we have drawn the graphs of three different functions. The values of the argument $x \in D$ and the corresponding values of the function y are respectively the abscissa and the ordinate of the points on the graph of this function.

From the definition of function it follows that to each value of $x \in D$ there corresponds exactly one value of y . Therefore, on the graph of a function there cannot be more than one point with the same abscissa; i.e., the graph cannot look like Fig. 11.

T1 Vertical Line Test. Every line parallel to the ordinate axis intersects the graph of a function in **at most one** point.

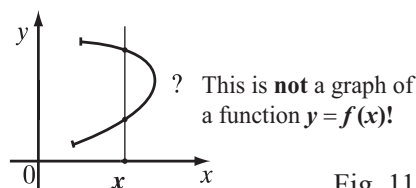


Fig. 11

Method of addition

When solving systems with **the method of addition**, we transform the given system into an equivalent one so that one of the resulting equations has only one unknown. In this method, we use the property of equalities:

If we add (subtract) term-by-term two true numerical equalities, we obtain a true numerical equality.



$$\begin{array}{r|l} 3 = 3 & + \\ 2 = 2 & \\ \hline 3 + 2 = 3 + 2 & \\ 5 = 5 & \end{array} \quad \begin{array}{r|l} 3 = 3 & - \\ 2 = 2 & \\ \hline 3 - 2 = 3 - 2 & \\ 1 = 1 & \end{array}$$

System (1) $\begin{cases} x + y = 3 \\ -x + y = -1 \end{cases}$ can be solved by substitution. We obtain $x = 2$ and $y = 1$.

If we add (subtract) the two equations of the system term-by-term, we obtain the systems

$$(2) \quad \begin{cases} 2y = 2 \\ x + y = 3 \end{cases} \quad \text{or} \quad \begin{cases} 2y = 2 \\ -x + y = -1 \end{cases}$$

$$(3) \quad \begin{cases} 2x = 4 \\ x + y = 3 \end{cases} \quad \text{or} \quad \begin{cases} 2x = 4 \\ -x + y = -1 \end{cases}$$

When we substitute the resulting values (2, 1) for x and y into systems (2) and (3), we will establish that they are equivalent to system (1).

We obtain $(1) \Leftrightarrow (2)$ and $(1) \Leftrightarrow (3)$.

Example 1

Solve the system $\begin{cases} x + 3y = 6 \\ 2x - 7y = -1 \end{cases}$.

Solution:

By addition:

$$\begin{array}{r|l} x + 3y = 6 & \cdot (-2) \\ 2x - 7y = -1 & \\ \hline -2x - 6y = -12 & + \\ 2x - 7y = -1 & \\ \hline -6y - 7y = -12 - 1 & \\ x + 3y = 6 & \\ -13y = -13 & \quad x + 3y = 6 \\ y = 1 & \quad x + 3 \cdot 1 = 6 \\ & \quad x = 3 \end{array}$$

By subtraction:

$$\begin{array}{r|l} x + 3y = 6 & \cdot 2 \\ 2x - 7y = -1 & \\ \hline 2x + 6y = 12 & - \\ 2x - 7y = -1 & \\ \hline 6y - (-7y) = 12 - (-1) & \\ 2x - 7y = -1 & \\ 6y + 7y = 12 + 1 & \quad 2x - 7y = -1 \\ y = 1 & \quad 2x - 7 \cdot 1 = -1 \\ & \quad x = 3 \end{array}$$

Answer: The solution to the system is (3, 1).

Addition algorithm

- We multiply one of the equations by a suitably chosen number so that the coefficients of one of the unknowns in the two equations of new system become:
 - opposite numbers $\rightarrow$ then we add the two equations term-by-term, or
 - equal numbers $\rightarrow$ then we subtract the two equations term-by-term.
- We obtain a system equivalent to the given in which one of the equations has only one unknown.
- We solve the equation with one unknown and find its solution (a.k.a. *root*).
- We substitute the resulting value in any of the equations of the given system and find the value of the other unknown.

Example 2

Solve the systems: a) $\begin{cases} 2(2x+5) - 3(y-1) = 12 \\ \frac{x+7}{4} + \frac{1}{3}(y+2) = 6; \end{cases}$ b) $\begin{cases} \frac{x-y}{2} - \frac{2x-1}{4} = y \\ \frac{3x+5y}{7} - y = 1. \end{cases}$

Solution:

$$\begin{array}{l} \text{a) } \begin{cases} 2(2x+5) - 3(y-1) = 12 \\ 3(x+7) + 4(y+2) = 72 \\ \dots\dots\dots \\ 4x - 3y = -1 \quad | \cdot 3 \\ 3x + 4y = 43 \quad | \cdot (-4) \\ \hline 12x - 9y = -3 \\ -12x - 16y = -172 \quad + \\ \hline -9y - 16y = -3 - 172 \\ -25y = -175 \\ y = 7 \\ 4x - 3 \cdot 7 = -1 \\ 4x - 21 = -1 \\ 4x = 20 \\ x = 5 \end{cases} \end{array}$$

Answer: The solution to the system is (5, 7).

$$\begin{array}{l} \text{b) } \begin{cases} \frac{x-y}{2} - \frac{2x-1}{4} = y \\ \frac{3x+5y}{7} - y = 1 \\ \dots\dots\dots \\ \cancel{2x} - 2y - \cancel{2x} + 1 = 4y \\ 3x + 5y - 7y = 7 \\ \hline 6y = 1 \rightarrow y = \frac{1}{6} \\ 3x - 2y = 7 \\ 3x - 2 \cdot \frac{1}{6} = 7 \\ 3x = 7 + \frac{1}{3} \\ 3x = \frac{22}{3} \\ x = \frac{22}{9} = 2\frac{4}{9} \end{cases} \end{array}$$

Answer: The solution to the system is $(2\frac{4}{9}, \frac{1}{6})$.

Example 3

Solve the system $\begin{cases} 2\frac{x+5}{7} + 3\frac{y-4}{2} = 7 \\ 3\frac{x+5}{7} - 5\frac{y-4}{2} = 1. \end{cases}$

Solution:

We substitute:

$$\begin{cases} \frac{x+5}{7} = u \\ \frac{y-4}{2} = v. \end{cases}$$

We obtain:

$$\begin{cases} 2u + 3v = 7 \quad | \cdot 3 \\ 3u - 5v = 1 \quad | \cdot (-2) \\ \dots\dots\dots \\ 19v = 19, v = 1 \\ 2u + 3 \cdot 1 = 7, u = 2. \end{cases}$$

$$\begin{cases} \frac{x+5}{7} = 2 \\ \frac{y-4}{2} = 1 \\ \hline x+5=14 \\ y-4=2 \\ \hline x=9, y=6 \end{cases}$$

Answer: The solution to the system is (9, 6).

Exercises

Solve by adding the systems:

$$\boxed{1} \quad \begin{cases} 3x - y = 18 \\ 5x + y = 22. \end{cases}$$

$$\boxed{2} \quad \begin{cases} y - x = 5 \\ x + 2y = 3. \end{cases}$$

$$\boxed{3} \quad \begin{cases} x + 5y = 7 \\ x - 3y = -1. \end{cases}$$

$$\boxed{4} \quad \begin{cases} 2x + 3y = 26 \\ 4x - y = 24. \end{cases}$$

$$\boxed{5} \quad \begin{cases} 7x - 3y = 15 \\ 5x + 6y = 27. \end{cases}$$

$$\boxed{6} \quad \begin{cases} 5x + 3y = 44 \\ 2x + 4y = 12. \end{cases}$$

$$\boxed{7} \quad \begin{cases} 7x - 3y + 1 = 0 \\ 4x - 5y + 17 = 0. \end{cases}$$

$$\boxed{8} \quad \begin{cases} 25x - 4y + 1 = 0 \\ 31x - 5y + 16 = 0. \end{cases}$$