

College Mathematics

Third Edition

Development Team

Tracey Haynie

Jenifer Bohart

Carla Stroud

Amy Volpe

Portions of this textbook adapted from the following OER sources:

Dimensional Analysis, Percentages, Savings and Loans, Sets, Probability, Statistics, Describing Data — “Math in Society” by David Lippman

Percentages – “Basic Arithmetic Student Workbook, 2nd Ed.” by Scottsdale Community College

Equations, Linear Equations – “Introductory Algebra Student Workbook, 7th Ed.” by Scottsdale Community College

Equations, Linear Equations, Exponential Equations, Regression – “Intermediate Algebra Student Workbook, 4th Ed.” by Scottsdale Community College

Measurement — The NROC Project,

http://nrocmath.org/products/higher-ed/developmental-mathematics/#contents_tab

Statistics Practice Problems — “Introductory Statistics” by OpenStax College

<https://openstax.org/details/introductory-statistics>



College Mathematics by Scottsdale Community College is licensed under a [Creative Commons Attribution–NonCommercial–ShareAlike 4.0 Unported License](https://creativecommons.org/licenses/by-nc-sa/4.0/).

Table of Contents

CHAPTER 1 – MEASUREMENT AND DIMENSIONAL ANALYSIS	1
SECTION 1.1: LENGTH	2
SECTION 1.2: WEIGHT	6
SECTION 1.3: CAPACITY	9
SECTION 1.4: THE METRIC SYSTEM	12
SECTION 1.5: CONVERTING BETWEEN SYSTEMS.....	18
SECTION 1.6: PROBLEM SOLVING USING RATES AND DIMENSIONAL ANALYSIS	22
CHAPTER 1 – PRACTICE PROBLEMS	28
CHAPTER 2 – EQUATIONS AND LINEAR BEHAVIOR.....	32
SECTION 2.1: SOLVING EQUATIONS	33
SECTION 2.2: APPLICATIONS OF SOLVING EQUATIONS.....	36
SECTION 2.3: LINEAR EQUATIONS	37
CHAPTER 2 – PRACTICE PROBLEMS	48
CHAPTER 3 – PERCENTAGES	52
SECTION 3.1: PERCENTAGE BASICS.....	53
SECTION 3.2: APPLICATIONS INVOLVING PERCENTAGES	54
SECTION 3.3: ABSOLUTE AND RELATIVE CHANGE	56
SECTION 3.4: PERCENT INCREASE AND PERCENT DECREASE	59
SECTION 3.5: EXPONENTIAL EQUATIONS.....	61
CHAPTER 3 – PRACTICE PROBLEMS	69
CHAPTER 4 – MODELING.....	75
SECTION 4.1: LINEAR REGRESSION	76
SECTION 4.2: CORRELATION.....	83
SECTION 4.3: EXPONENTIAL REGRESSION	86
SECTION 4.4: MODELING APPLICATIONS	90
CHAPTER 4 – PRACTICE PROBLEMS	96
CHAPTER 5 – SIMPLE AND COMPOUND INTEREST.....	103
SECTION 5.1: SIMPLE INTEREST.....	104
SECTION 5.2: COMPOUND INTEREST	106
SECTION 5.3: ANNUAL PERCENTAGE YIELD (APY).....	110
CHAPTER 5 – PRACTICE PROBLEMS	113
CHAPTER 6 – ANNUITIES AND LOANS.....	116
SECTION 6.1: ANNUITIES.....	117
SECTION 6.2: PAYOUT ANNUITIES	123
SECTION 6.3: LOANS	126
CHAPTER 6 – PRACTICE PROBLEMS	132
CHAPTER 7 – SETS AND VENN DIAGRAMS	136
SECTION 7.1: SET BASICS	137
SECTION 7.2: UNION, INTERSECTION, AND COMPLEMENT	138

SECTION 7.3: VENN DIAGRAMS	141
SECTION 7.4: CARDINALITY	142
CHAPTER 7 – PRACTICE PROBLEMS	152
CHAPTER 8 – INTRODUCTION TO PROBABILITY.....	159
SECTION 8.1: TYPES OF PROBABILITY	160
SECTION 8.2: BASIC CONCEPTS	161
SECTION 8.3: COMPLEMENTARY EVENTS	164
SECTION 8.4: EXPECTED VALUE	165
CHAPTER 8 – PRACTICE PROBLEMS	169
CHAPTER 9 – PROBABILITY.....	172
SECTION 9.1: INDEPENDENT EVENTS	173
SECTION 9.2: CONDITIONAL PROBABILITY.....	175
SECTION 9.3: “OR” PROBABILITIES.....	177
CHAPTER 9 – PRACTICE PROBLEMS	181
CHAPTER 10 – COUNTING.....	187
SECTION 10.1: BASIC COUNTING	188
SECTION 10.2: PERMUTATIONS.....	191
SECTION 10.3: COMBINATIONS	194
SECTION 10.4: FINDING PROBABILITIES USING COUNTING	196
CHAPTER 10 – PRACTICE PROBLEMS	201
CHAPTER 11 – DESCRIBING DATA	204
SECTION 11.1: PRESENTING CATEGORICAL DATA GRAPHICALLY.....	205
SECTION 11.2: PRESENTING QUANTITATIVE DATA GRAPHICALLY	211
SECTION 11.3: DESCRIBING A DISTRIBUTION	216
SECTION 11.4: MEASURES OF CENTRAL TENDENCY	220
SECTION 11.5: MEASURES OF VARIATION	225
SECTION 11.6: QUARTILES, FIVE NUMBER SUMMARY, AND BOXPLOTS.....	229
CHAPTER 11 – PRACTICE PROBLEMS	237
CHAPTER 12 – THE NORMAL DISTRIBUTION.....	249
SECTION 12.1: THE NORMAL DISTRIBUTION	250
SECTION 12.2: THE EMPIRICAL RULE.....	251
SECTION 12.3: STANDARD SCORES (Z – SCORES).....	254
SECTION 12.4: FINDING PERCENTAGES WITH THE NORMAL DISTRIBUTION	257
CHAPTER 12 – PRACTICE PROBLEMS	266
APPENDIX A: ANSWERS TO PRACTICE PROBLEMS	272

CHAPTER 1 – MEASUREMENT AND DIMENSIONAL ANALYSIS

INTRODUCTION

Measurement is a number that describes the size or amount of something. You can measure many things like length, area, capacity, weight, temperature and time.

At times, you may need to convert between units of measurement. For example, you might want to express your height using feet and inches (5 feet 4 inches) or using only inches (64 inches). This process of converting from one unit to another unit is called **unit analysis or dimensional analysis**.

Chapter 1 Learning Objectives:

- 1) Use dimensional analysis to convert between different units of measurement for length, mass, and capacity
- 2) Use dimensional analysis to convert between units of measurement in the metric system
- 3) Use dimensional analysis to convert between the metric and standard systems of measurement
- 4) Use rates and unit rates to problem solve

Sections

Section 1.1 – Length

Section 1.2 - Weight

Section 1.3 - Capacity

Section 1.4 - The Metric System

Section 1.5 - Converting Between Systems

Section 1.6 – Problem Solving Using Rates and Dimensional Analysis

SECTION 1.1: LENGTH

Length is the distance from one end of an object to the other end, or from one object to another. For example, the length of a letter-sized piece of paper is 11 inches. The system for measuring length in the United States is based on the four customary units of length: ***inch, foot, yard, and mile***. You can use any of these four U.S. customary measurement units to describe the length of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the length of a rug in feet rather than miles, and to describe a marathon in miles rather than inches.

The table below shows equivalents and conversion factors for the four customary units of measurement of length.

<u>Unit Equivalents</u>	<u>Conversion Factor</u> (longer to shorter)	<u>Conversion Factor</u> (shorter to longer)
1) 1 foot = 12 inches	$\frac{12 \text{ inches}}{1 \text{ foot}}$	$\frac{1 \text{ foot}}{12 \text{ inches}}$
2) 1 yard = 3 feet	$\frac{3 \text{ feet}}{1 \text{ yard}}$	$\frac{1 \text{ yard}}{3 \text{ feet}}$
3) 1 mile = 5280 feet	$\frac{5280 \text{ feet}}{1 \text{ mile}}$	$\frac{1 \text{ mile}}{5280 \text{ feet}}$

You can use the conversion factors to convert a measurement, such as feet, to another type of measurement, such as inches. Note that each of these conversion factors is a ratio of equal values, so each conversion factor equals 1. Multiplying a measurement by a conversion factor does not change the size of the measurement at all since it is the same as multiplying by 1; it just changes the units that you are using to measure.

There are many more inches for a measurement than there are feet for the same measurement, as feet is a longer unit of measurement. So, you could use the conversion factor $\frac{12 \text{ inches}}{1 \text{ foot}}$.

If a length is measured in feet, and you'd like to convert the length to yards, you can think, "I am converting from a shorter unit to a longer one, so the length in yards will be less than the length in feet." You could use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$.

Dimensional Analysis: The Factor-Label Method

You can use the **factor label method** to convert a length from one unit of measure to another using the conversion factors. In the factor label method, you multiply by unit fractions to convert a measurement from one unit to another. Study the example below to see how the factor label method can be used to convert a measurement given in feet into an equivalent number of inches.

Example 1: How many inches are in $3\frac{1}{2}$ feet?

Solution: Begin by reasoning about your answer. Since a foot is longer than an inch, this means the answer would be greater than $3\frac{1}{2}$ feet. Now state your givens and your goal.

Given: $3\frac{1}{2}$ feet

Goal: Convert to inches, or equivalently, solve $3\frac{1}{2}$ feet = _____ inches

1) Find the conversion factor that compares inches and feet, with "inches" in the numerator, and multiply.

$$3\frac{1}{2} \text{ feet} \times \frac{12 \text{ inches}}{1 \text{ foot}} = \text{_____ inches}$$

2) Rewrite the mixed number as an improper fraction before multiplying.

$$\frac{7 \text{ feet}}{2} \times \frac{12 \text{ inches}}{1 \text{ foot}} = \text{_____ inches} \quad \left(\text{Since } 3\frac{1}{2} = \frac{7}{2} \right)$$

3) You can cancel similar units when they appear in the numerator *and* the denominator. So here, cancel the similar units "feet" and "foot." This eliminates this unit from the problem.

$$\frac{7 \cancel{\text{ feet}}}{2} \times \frac{12 \text{ inches}}{1 \cancel{\text{ foot}}} = \frac{\cancel{7} \cancel{\text{ feet}}}{2} \times \frac{12 \text{ inches}}{1 \cancel{\text{ foot}}} = \frac{7}{2} \times \frac{12 \text{ inches}}{1}$$

4) Rewrite as multiplication of numerators and denominators. Then divide.

$$\frac{7}{2} \times \frac{12 \text{ inches}}{1} = \frac{7 \times 12 \text{ inches}}{2 \times 1} = \frac{84}{2} \text{ inches} = 42 \text{ inches}$$

Answer: $3\frac{1}{2}$ feet equals 42 inches.

Notice that by using the factor label method you can cancel the units out of the problem, just as if they were numbers. You can only cancel if the unit being cancelled is in both the numerator and denominator of the fractions you are multiplying. In the problem above, you cancelled *feet* and *foot* leaving you with *inches*, which is what you were trying to find.

What if you had used the wrong conversion factor as shown below?

$$\frac{7 \text{ feet}}{2} \times \frac{1 \text{ foot}}{12 \text{ inches}} = ?$$

You could not cancel the feet because the unit is not the same in *both* the numerator and the denominator. So if you complete the computation, you would still have both feet and inches in the answer and no conversion would take place.

Example 2: An interior decorator needs border trim for a home she is wallpapering. She needs 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room. How many yards of border trim does she need?

Solution: State your givens and your goal.

Given: 15 feet of border trim for the living room, 30 feet of border trim for the bedroom, and 26 feet of border trim for the dining room

Goal: How many yards of border trim does she need?

1) You need to find the total length of border trim that is needed for all three rooms in the house. Since the measurements for each room are given in feet, you can add the numbers.

$$15 \text{ feet} + 30 \text{ feet} + 26 \text{ feet} = 71 \text{ feet}$$

2) Observe that our goal is the number of yards, not the number of feet. So we need to convert. How many yards is 71 feet? (Reason about the size of your answer. Since a yard is longer than a foot, there will be fewer yards. Expect your answer to be less than 71.)

$$71 \text{ feet} = \underline{\hspace{2cm}} \text{ yards}$$

3) Use the conversion factor $\frac{1 \text{ yard}}{3 \text{ feet}}$

$$\frac{71 \text{ feet}}{1} \times \frac{1 \text{ yard}}{3 \text{ feet}} = \underline{\hspace{2cm}} \text{ yards}$$

4) Since "feet" is in the numerator and denominator, you can cancel this unit. Then rewrite as multiplication of numerators and denominators.

$$\frac{71 \cancel{\text{feet}}}{1} \times \frac{1 \text{ yard}}{3 \cancel{\text{feet}}} = \frac{71 \cancel{\text{feet}}}{1} \times \frac{1 \text{ yard}}{3 \cancel{\text{feet}}} = \frac{71 \times 1 \text{ yard}}{1 \times 3}$$

5) Multiply and then rewrite the improper fraction as a mixed number.

$$\frac{71 \times 1 \text{ yard}}{1 \times 3} = \frac{71}{3} \text{ yards} = 23 \frac{2}{3} \text{ yards}$$

Answer: The interior designer needs $23 \frac{2}{3} \text{ yards}$ of border trim.

At times, you will need to make unit conversions for measurements in two or three dimensions (think finding an area of a field, or the volume of a container). The following is an example for converting units in two dimensions using an area problem.

Example 3: A landscaper is installing artificial turf for a football field. The dimensions of the football field are 120 yards long by 50 yards wide. His supplier sells artificial turf by the square foot. How many square feet of turf is needed for the field?

Solution: State your givens and your goal.

Given: The dimensions of the football field are 120 yards long by 50 yards wide.

Goal: How many square feet of turf is needed for the field?

1) You need to find the total area of the field that needs to be covered. Since both length and width are given in yards, you can multiply. (Note: The resulting units are in **square yards**.)

$$120 \text{ yards} \times 50 \text{ yards} = 6,000 \text{ square yards}$$

2) We are asked to find the number of square feet needed. So convert square yards to square feet. (Reason about the size of your answer. Since a square foot is smaller than a square yard, there will be more square feet. Expect your answer to be larger than 6000.)

$$\frac{6000 \text{ yd}^2}{1} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} = \frac{6000 \cancel{\text{yd}} \times \cancel{\text{yd}}}{1} \times \frac{3 \text{ ft}}{1 \cancel{\text{yd}}} \times \frac{3 \text{ ft}}{1 \cancel{\text{yd}}} = \frac{6000 \times 3 \text{ ft} \times 3 \text{ ft}}{1 \times 1 \times 1} = 54,000 \text{ ft}^2$$

Answer: The landscaper needs 54,000 square feet of artificial turf for the football field.

Section 1.1 You Try Problems:

- a) Use the Factor-Label Method to determine the number of feet in $2\frac{1}{2}$ miles.
- b) A fence company is measuring a rectangular area in order to install a fence around its perimeter. If the length of the rectangular area is 130 yards and the width is 75 feet, what is the total length of the distance to be fenced?

SECTION 1.2: WEIGHT

You often use the word **weight** to describe how heavy or light an object or person is. Weight is measured in the U.S. customary system using three units: ounces, pounds, and tons. An **ounce** is the smallest unit for measuring weight, a **pound** is a larger unit, and a **ton** is the largest unit. You can use any of the customary measurement units to describe the weight of something, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the weight of a human being in pounds rather than tons. It makes more sense to describe the weight of a car in tons rather than ounces.

The following table shows the unit conversions and conversion factors that are used to make conversions between customary units of weight.

<u>Unit Equivalents</u>	<u>Conversion Factor</u> (heavier to lighter)	<u>Conversion Factor</u> (lighter to heavier)
1) 1 pound = 16 ounces	$\frac{16 \text{ ounces}}{1 \text{ pound}}$	$\frac{1 \text{ pound}}{16 \text{ ounces}}$
2) 1 ton = 2000 pounds	$\frac{2000 \text{ pounds}}{1 \text{ ton}}$	$\frac{1 \text{ ton}}{2000 \text{ pounds}}$

Example 4: A Use the Factor Label Method to determine the number of ounces in $2\frac{1}{4}$ pounds.

Solution: State your givens and your goal.

Given: $2\frac{1}{4}$ pounds

Goal: Determine the equivalent number of ounces

1) Begin by reasoning about your answer. Since a pound is heavier than an ounce, expect your answer to be a number greater than $2\frac{1}{4}$.

$$2\frac{1}{4} \text{ pounds} = \underline{\hspace{2cm}} \text{ ounces}$$

2) Multiply by the conversion factor that relates ounces and pounds: $\frac{16 \text{ ounces}}{1 \text{ pound}}$

$$2\frac{1}{4} \text{ pounds} \times \frac{16 \text{ ounces}}{1 \text{ pound}} = \underline{\hspace{2cm}} \text{ ounces}$$

3) Write the mixed number as an improper fraction. The common unit "pound" can be cancelled because it appears in both the numerator and denominator.

$$\frac{9\cancel{\text{pounds}}}{4} \times \frac{16 \text{ ounces}}{1\cancel{\text{pound}}} = \underline{\hspace{2cm}} \text{ ounces}$$

4) Multiply and simplify.

$$\frac{9}{4} \times \frac{16 \text{ ounces}}{1} = \frac{9 \times 16 \text{ ounces}}{4 \times 1} = \frac{144 \text{ ounces}}{4} = 36 \text{ ounces}$$

Answer: There are 36 ounces in $2\frac{1}{4}$ pounds.

There are times when you need to perform calculations on measurements that are given in different units. To solve these problems, you need to convert one of the measurements to the same unit of measurement as the other measurement. Think about whether the unit you are converting to is smaller or larger than the unit you are converting from. This will help you be sure that you are making the right computation. You can use the factor label method to make the conversion from one unit to another.

The following examples require converting between units of weight.

Example 5: A municipal trash facility allows a person to throw away a maximum of 30 pounds of trash per week. Last week, 140 people threw away the maximum allowable trash. How many tons of trash did this equal?

Solution: State your givens and your goal.

Given: maximum of 30 pounds of trash per week per person, 140 people threw away the maximum allowable trash

Goal: How many tons of trash did this equal?

1) Determine the total trash for the week expressed in pounds. If 140 people each throw away 30 pounds, you can find the total by multiplying.

$$140 \times 30 \text{ pounds} = 4,200 \text{ pounds}$$

2) Then convert 4,200 pounds to tons. Reason about your answer. Since a ton is heavier than a pound, expect your answer to be a number less than 4,200.

$$4,200 \text{ pounds} = \underline{\hspace{2cm}} \text{ tons}$$

3) Find the conversion factor appropriate for the situation: $\frac{1 \text{ ton}}{2000 \text{ pounds}}$

$$\frac{4200 \cancel{\text{ pounds}}}{1} \times \frac{1 \text{ ton}}{2000 \cancel{\text{ pounds}}} = \underline{\hspace{2cm}} \text{ tons}$$

4) Multiply and simplify. Rewrite your result as a mixed number.

$$\frac{4200}{1} \times \frac{1 \text{ ton}}{2000} = \frac{4200 \times 1 \text{ ton}}{1 \times 2000} = \frac{4200}{2000} \text{ tons} = 2 \frac{1}{10} \text{ tons}$$

Answer: The total amount of trash generated is $2 \frac{1}{10} \text{ tons}$.

Example 6: The post office charges \$0.44 to mail something that weighs an ounce or less. The charge for each additional ounce, or fraction of an ounce, of weight is \$0.17. At this rate, how much will it cost to mail a package that weighs 2 pounds 3 ounces?

Solution: State your givens and your goal.

Given: \$0.44 to mail something that weighs an ounce or less, each additional ounce of weight is \$0.17

Goal: How much will it cost to mail a package that weighs 2 pounds 3 ounces?

1) Since the pricing is for ounces, convert the weight of the package from pounds and ounces into just ounces.

$$2 \text{ pounds } 3 \text{ ounces} = \underline{\hspace{2cm}} \text{ ounces}$$

2) First use the factor label method to convert 2 pounds to ounces.

$$\frac{2 \cancel{\text{ pounds}}}{1} \times \frac{16 \text{ oz}}{1 \cancel{\text{ pound}}} = \frac{2 \times 16}{1} \text{ oz} = 32 \text{ oz}$$

3) Add the additional 3 ounces to find the weight of the package.

$$2 \text{ pounds } 3 \text{ ounces} = 32 \text{ ounces} + 3 \text{ ounces} = 35 \text{ ounces}$$

4) Apply the pricing formula. \$0.44 for the first ounce and \$0.17 for each additional ounce.

$$\$0.44 + \$0.17(34) = \$0.44 + \$5.78 = \$6.22$$

Answer: It will cost \$6.22 to mail a package that weighs 2 pounds 3 ounces.

Section 1.2 You Try Problems:

a) How many pounds is 72 ounces?

b) The average weight of a northern Bluefin tuna is 1,800 pounds. The average weight of a great white shark is $2\frac{1}{2}$ tons. On average, how much more does a great white shark weigh, in pounds, than a northern Bluefin tuna?

SECTION 1.3: CAPACITY

Capacity is the amount of liquid (or other pourable substance) that an object can hold when it's full. When a liquid, such as milk, is being described in gallons or quarts, this is a measure of capacity.

There are five main units for measuring capacity in the U.S. customary measurement system. The smallest unit of measurement is a fluid ounce. "Ounce" is also used as a measure of weight, so it is important to use the word "fluid" with ounce when you are talking about capacity. Sometimes the prefix "fluid" is not used when it is clear from the context that the measurement is capacity, not weight.

The other units of capacity in the customary system are the cup, pint, quart, and gallon. The table below describes each unit of capacity and provides an example to illustrate the size of the unit of measurement.

You can use any of these five measurement units to describe the capacity of an object, but it makes more sense to use certain units for certain purposes. For example, it makes more sense to describe the capacity of a swimming pool in gallons and the capacity of an expensive perfume in fluid ounces.

The table below shows some of the most common equivalents and conversion factors for the five customary units of measurement of capacity.

<u>Unit Equivalents</u>	<u>Conversion Factor</u> (more to less)	<u>Conversion Factor</u> (less to more)
1) 1 pint = 2 cups	$\frac{1 \text{ pint}}{2 \text{ cups}}$	$\frac{2 \text{ cups}}{1 \text{ pint}}$
2) 1 quart = 2 pints	$\frac{1 \text{ quart}}{2 \text{ pints}}$	$\frac{2 \text{ pints}}{1 \text{ quart}}$
3) 1 quart = 4 cups	$\frac{1 \text{ quart}}{4 \text{ cups}}$	$\frac{4 \text{ cups}}{1 \text{ quart}}$

Unit Equivalents**Conversion Factor**

(more to less)

Conversion Factor

(less to more)

4) 1 gallon = 4 quarts

$$\frac{1 \text{ gallon}}{4 \text{ quarts}}$$

$$\frac{4 \text{ quarts}}{1 \text{ gallon}}$$

5) 1 gallon = 16 cups

$$\frac{1 \text{ gallon}}{16 \text{ cups}}$$

$$\frac{16 \text{ cups}}{1 \text{ gallon}}$$

As with converting units of length and weight, you can use the factor label method to convert from one unit of capacity to another.

Example 7: Use the Factor Label Method to determine the number of pints in $2\frac{3}{4}$ gallons.

Solution: State your givens and your goal.

Given: $2\frac{3}{4}$ gallons.

Goal: Find the equivalent amount in pints.

1) Begin by reasoning about your answer. Since a gallon is larger than a pint, expect the answer in pints to be a number greater than $2\frac{3}{4}$.

$$2\frac{3}{4} \text{ gallons} = \text{_____pints}$$

2) The table above does not contain a conversion factor for gallons and pints, so you cannot convert it in one step. However, you can use quarts as an intermediate unit, as shown here. Set up the equation so that two sets of labels cancel—gallons and quarts.

$$2\frac{3}{4} \text{ gallons} = \frac{11 \cancel{\text{gallons}}}{4} \times \frac{4 \cancel{\text{quarts}}}{1 \cancel{\text{gallon}}} \times \frac{2 \text{ pints}}{1 \cancel{\text{quart}}} = \frac{11 \times 4 \times 2}{4 \times 1 \times 1} \text{ pints} = 22 \text{ pints}$$

Answer: $2\frac{3}{4}$ gallons is equivalent to 22 pints.

Another way to work the problem above would be to first change 1 gallon to 16 cups and change 2 quarts to 8 cups. Then add: $16 + 8 = 24$ cups.

Example 8: Natasha is making lemonade to bring to the beach. She has two containers. One holds one gallon and the other holds 2 quarts. If she fills both containers, how many cups of lemonade will she have?

Solution: State your givens and your goal.

Given: Two containers for lemonade; one holds one gallon and the other holds 2 quarts, fills both containers

Goal: How many cups of lemonade will she have?

1) This problem requires you to find the sum of the capacity of each container and then convert that sum to cups.

$$1 \text{ gallon} + 2 \text{ quarts} = \underline{\hspace{1cm}} \text{ cups}$$

2) First, find the sum in quarts. 1 gallon is equal to 4 quarts.

$$4 \text{ quarts} + 2 \text{ quarts} = 6 \text{ quarts}$$

3) Since the problem asks for the capacity in cups, convert 6 quarts to cups.

$$\frac{6 \text{ quarts}}{1} \times \frac{2 \text{ pints}}{1 \text{ quart}} \times \frac{2 \text{ cups}}{1 \text{ pint}} = \underline{\hspace{1cm}} \text{ cups}$$

4) Cancel units that appear in both the numerator and denominator and multiply.

$$\frac{\cancel{6 \text{ quarts}}}{1} \times \frac{2 \cancel{\text{pints}}}{\cancel{1 \text{ quart}}} \times \frac{2 \text{ cups}}{1 \cancel{\text{pint}}} = \frac{6 \times 2 \times 2 \text{ cups}}{1 \times 1 \times 1} = 24 \text{ cups}$$

Answer: Natasha will have 24 cups of lemonade.

Section 1.3 You Try Problems:

Alan is making chili. He is using a recipe that makes 24 cups of chili. He has a 5-quart pot and a 2-gallon pot and is trying to determine whether the chili will all fit in one of these pots. Which of the pots will fit the chili?

- A) The chili will not fit into either of the pots.
- B) The chili can fit into either pot.
- C) The chili will fit into the 5-quart pot only.
- D) The chili will fit into the 2-gallon pot only.

SECTION 1.4: THE METRIC SYSTEM

In the United States, both the **U.S. customary measurement system** and the **metric system** are used, especially in medical, scientific, and technical fields. In most other countries, the metric system is the primary system of measurement. If you travel to other countries, you will see that road signs list distances in kilometers and milk is sold in liters. People in many countries use words like “kilometer,” “liter,” and “milligram” to measure the length, volume, and weight of different objects. These measurement units are part of the metric system.

Unlike the U.S. customary system of measurement, the metric system is based on 10s. For example, a liter is 10 times larger than a deciliter, and a centigram is 10 times larger than a milligram. This idea of “10” is not present in the U.S. customary system—there are 12 inches in a foot, and 3 feet in a yard...and 5,280 feet in a mile!

What if you have to find out how many milligrams are in a decigram? Or, what if you want to convert meters to kilometers? Understanding how the metric system works is a good start.

What is Metric?

The metric system uses units such as meter, gram, and liter to measure length, mass, and liquid volume (capacity), just as the U.S. customary system uses feet, ounces, and quarts to measure these.

In addition to the difference in the basic units, the metric system is based on 10s, and different measures for length include kilometer, meter, decimeter, centimeter, and millimeter. Notice that the word “meter” is part of all of these units.

The metric system also applies the idea that units within the system get larger or smaller by a power of 10. This means that a meter is 100 times larger than a centimeter, and a kilogram is 1,000 times heavier than a gram. You will explore this idea a bit later. For now, notice how this idea of “getting bigger or smaller by 10” is very different than the relationship between units in the U.S. customary system, where 3 feet equals 1 yard, and 16 ounces equals 1 pound.

Length, Mass, and Volume

The table below shows the basic units of the metric system. Note that the names of all metric units follow from these three basic units.

Base Units of Metric System

Length
Meter

Mass
gram

Volume
liter

Other Units You May See

kilometer	kilogram	dekaliter
centimeter	centigram	centiliter
millimeter	milligram	milliliter

In the metric system, the basic unit of length is the meter. A meter is slightly larger than a yardstick, or just over three feet.

The basic metric unit of mass is the gram. A regular-sized paperclip has a mass of about 1 gram. Among scientists, one gram is defined as the mass of water that would fill a 1-centimeter cube. You may notice that the word “mass” is used here instead of “weight.” In the sciences and technical fields, a distinction is made between weight and mass. Weight is a measure of the pull of gravity on an object. For this reason, an object’s weight would be different if it was weighed on Earth or on the moon because of the difference in the gravitational forces. However, the object’s mass would remain the same in both places because mass measures the amount of substance in an object. As long as you are planning on only measuring objects on Earth, you can use mass/weight fairly interchangeably—but it is worth noting that there is a difference!

Finally, the basic metric unit of volume is the liter. A liter is slightly larger than a quart.

 The handle of a shovel is about 1 meter.	 A paperclip weighs about 1 gram.	 A medium-sized container of milk is about 1 liter.
---	---	---

Prefixes in the Metric System

The metric system is a base 10 system. This means that each successive unit is 10 times larger than the previous one.

The names of metric units are formed by adding a prefix to the basic unit of measurement. To tell how large or small a unit is, you look at the **prefix**. To tell whether the unit is measuring length, mass, or volume, you look at the base.

Prefixes in the Metric System

<i>kilo-</i>	<i>hecto-</i>	<i>deka-</i>	meter gram liter	<i>deci-</i>	<i>centi-</i>	<i>milli-</i>
1,000 times larger than base unit	100 times larger than base unit	10 times larger than base unit	base units	10 times smaller than base unit	100 times smaller than base unit	1,000 times smaller than base unit

Using this table as a reference, you can see the following:

- A kilogram is 1,000 times larger than one gram (so 1 kilogram = 1,000 grams).
- A centimeter is 100 times smaller than one meter (so 1 meter = 100 centimeters).
- A dekaliter is 10 times larger than one liter (so 1 dekaliter = 10 liters).

Here is a similar table that just shows the metric units of measurement for mass, along with their size relative to 1 gram (the base unit). The common abbreviations for these metric units have been included as well.

Measuring Mass in the Metric System

kilogram (kg)	hectogram (hg)	dekagram (dag)	gram (g)	decigram (dg)	centigram (cg)	milligram (mg)
1,000 grams	100 grams	10 grams	1 gram	0.1 gram	0.01 gram	0.001 gram

Since the prefixes remain constant through the metric system, you could create similar charts for length and volume. The prefixes have the same meanings whether they are attached to the units of length (meter), mass (gram), or volume (liter).

Converting Within the Metric System

While knowing the different units used in the metric system is important, the real purpose behind learning the metric system is for you to be able to use these measurement units to calculate the size, mass, or volume of different objects. In practice, it is often necessary to convert one metric measurement to another unit—this happens frequently in the medical, scientific, and technical fields, where the metric system is commonly used.

The tables below show some of the **unit equivalents** for length, mass, and volume in the metric system.

Common Metric Length Conversions

1 centimeter = 10 millimeters	1 cm = 10 mm
1 decimeter = 10 centimeters	1 dm = 10 cm
1 meter = 100 centimeters	1 m = 100 cm
1 kilometer = 1000 meters	1 km = 1000 m

Common Metric Mass Conversions

1 gram = 1000 milligrams	1 g = 1000 mg
1 decagram = 10 grams	1 dag = 10 g
1 kilogram = 1000 grams	1 kg = 1000 g

Common Metric Volume Conversions

1 centiliter = 10 milliliters	1 cl = 10 ml
1 deciliter = 10 centiliters	1 dl = 10 cl
1 liter = 1000 milliliters	1 l = 1000 ml
1 liter = 10 deciliters	1 l = 10 dl
1 kiloliter = 1000 liters	1 kl = 1000 l

Remember that the metric system is based on the notion that each unit is 10 times larger than the one that came before it.

Example 9: Convert 7,225 centimeters to meters.

Solution: State your givens and your goal.

Given: 7,225 centimeters

Goal: Find the equivalent length in meters

1) Meters is larger than centimeters, so you expect your answer to be less than 7,225.

$$7,225 \text{ cm} = \underline{\hspace{2cm}} \text{ m}$$

2) Using the factor label method, write 7,225 cm as a fraction and use unit fractions to convert it to meters.

$$\frac{7,225 \text{ cm}}{1} \times \frac{1 \text{ m}}{100 \text{ cm}} = \underline{\hspace{2cm}} \text{ m}$$

3) Cancel similar units, multiply, and simplify.

$$\frac{7,225\cancel{\text{ cm}}}{1} \times \frac{1\cancel{\text{ m}}}{100\cancel{\text{ cm}}} = \frac{7225 \times 1\cancel{\text{ m}}}{1 \times 100} = 72.25\cancel{\text{ m}}$$

Answer: 7,225 centimeters = 72.25 m

Learning how to solve real-world problems using metric conversions is as important as learning how to do the conversions themselves. Mathematicians, scientists, nurses, and even athletes are often confronted with situations where they are presented with information using metric measurements, and must then make informed decisions based on that data.

To solve these problems effectively, you need to understand the context of a problem, perform conversions, and then check the reasonableness of your answer. Do all three of these steps and you will succeed in whatever measurement system you find yourself using.

Example 10: In the Summer Olympic Games, athletes compete in races of the following lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters. If a runner were to run in all these races, how many kilometers would he run?

Solution: State your givens and your goal.

Given: The lengths: 100 meters, 200 meters, 400 meters, 800 meters, 1500 meters, 5000 meters and 10,000 meters.

Goal: If a runner were to run in all these races, how many kilometers would he run?

1) To figure out how many kilometers he would run, you need to first add all of the lengths of the races together and *then* convert that measurement to kilometers.

$$100\text{ m} + 200\text{ m} + 400\text{ m} + 800\text{ m} + 1500\text{ m} + 5000\text{ m} + 10,000\text{ m} = 18,000\text{ m}$$

2) Use the factor label method and unit fractions to convert from meters to kilometers.

$$\frac{18,000\cancel{\text{ m}}}{1} \times \frac{1\cancel{\text{ km}}}{1000\cancel{\text{ m}}} = \underline{\hspace{2cm}}\text{ km}$$

3) Cancel common units, multiply, and divide.

$$\frac{18,000\cancel{\text{ m}}}{1} \times \frac{1\cancel{\text{ km}}}{1000\cancel{\text{ m}}} = \frac{18,000 \times 1\cancel{\text{ km}}}{1 \times 1000} = 18\cancel{\text{ km}}$$

Answer: The runner would run 18 km

Example 11: One bottle holds 295 dL while another one holds 28,000 mL. What is the difference in capacity between the two bottles?

Solution: State your givens and your goal.

Given: Two bottle capacities: 295 dL and 28,000 mL.

Goal: What is the difference in capacity between the two bottles?

1) The two measurements are in different units. You can convert both units to liters and then compare them.

$$295 \text{ dL} = \underline{\hspace{2cm}} \text{ L} \qquad \frac{295 \text{ dL}}{1} \times \frac{1 \text{ L}}{10 \text{ dL}} = \frac{295 \text{ L}}{10} = 29.5 \text{ L}$$

$$28,000 \text{ mL} = \underline{\hspace{2cm}} \text{ L} \qquad \frac{28,000 \text{ mL}}{1} \times \frac{1 \text{ L}}{1000 \text{ mL}} = \frac{28,000 \text{ L}}{1000} = 28 \text{ L}$$

2) The question asks for “difference in capacity” between the bottles so we will subtract.

$$29.5 \text{ L} - 28 \text{ L} = 1.5 \text{ L}$$

Answer: There is a difference in capacity of 1.5 liters between the two bottles.

Example 12: A two-liter bottle contains 87 centiliters of oil and 4.1 deciliters of water. How much more liquid is needed to fill the bottle?

Solution: State your givens and your goal.

Given: a 2-Liter bottle contains 87 cL of oil and 4.1 dL of water.

Goal: How much more liquid is needed to fill the bottle?

1) You are looking for the amount of liquid needed to fill the bottle. First determine the amount of liquid already in the bottle by convert both measurements to liters and adding.

$$87 \text{ cL} = \underline{\hspace{2cm}} \text{ L} \qquad \frac{87 \text{ cL}}{1} \times \frac{1 \text{ L}}{100 \text{ cL}} = \frac{87 \text{ L}}{100} = 0.87 \text{ L}$$

$$4.1 \text{ dL} = \underline{\hspace{2cm}} \text{ L} \qquad \frac{4.1 \text{ dL}}{1} \times \frac{1 \text{ L}}{10 \text{ dL}} = \frac{4.1 \text{ L}}{10} = 0.41 \text{ L}$$

$$\text{Total liquid in bottle: } 0.87 \text{ L} + 0.41 \text{ L} = 1.28 \text{ L}$$

- 2) Find the difference between 2 L and the amount of liquid in the bottle to see how much more liquid is needed to fill the bottle.

$$2 \text{ L} - 1.28 \text{ L} = 0.72 \text{ L}$$

Answer: The amount of liquid needed to fill the bottle is 0.72 liters.

Section 1.4 You Try Problems:

- a) The distance from Scottsdale to Las Vegas is 421 km.

What is the equivalent distance in meters?

What is the equivalent distance in centimeters?

- b) One boxer weighs in at 85 kg. He is 80 dekagrams heavier than his opponent. How much does his opponent weigh?

SECTION 1.5: CONVERTING BETWEEN SYSTEMS

We have spent the last several sections learning about the U.S. customary system of measurement, and the metric system. As you might guess, there are many applications where it is useful to be able to convert between measurements of length, mass, and volume (capacity) in the two systems. The table below gives some useful conversions between U.S. and metric measurements:

Approximate Conversions between Customary and Metric Systems

Length	Mass	Volume
1 inch = 2.540 centimeters	1 pound = 0.4536 kilograms	1 fluid ounce = 29.574 milliliters
1 foot = 0.3048 m	1 ounce = 28 grams	1 quart = 0.9464 liters
1 yard = 0.9144 meters		1 gallon = 3.785 liters
1 mile = 1.6093 kilometers		

Example 13: An Olympic sprinter competes in the 400m dash. How many yards is the race?

Solution: State your givens and your goal.

Given: 400m dash

Goal: How many yards is the race?

- 1) Reason about your answer. A yard is less than a meter, so you expect your answer to be more than 400.

$$400 \text{ m} = \underline{\hspace{2cm}} \text{ yd}$$

2) Using the factor label method, write 400 m as a fraction and use unit fractions to convert it to yards.

$$\frac{400 \text{ m}}{1} \times \frac{0.9144 \text{ yd}}{1 \text{ m}} = \underline{\hspace{2cm}} \text{ yd}$$

3) Cancel similar units, multiply, and simplify.

$$\frac{400 \cancel{\text{ m}}}{1} \times \frac{0.9144 \text{ yd}}{1 \cancel{\text{ m}}} = \frac{400 \times 0.9144 \text{ yd}}{1 \times 1} = 437.45 \text{ yd}$$

Answer: A 400 m dash is 437.45 yards.

Example 14: A patient must be weighed prior to surgery so that the proper dosage of anesthesia can be given. A nurse weighs a patient and finds that he weighs 205 pounds. The anesthesiologist prefers weights in kilograms. What is the patient's weight in kilograms? Round to the nearest hundredth.

Solution: State your givens and your goal.

Given: patient weighs 205 pounds

Goal: What is the patient's weight in kilograms?

1) Reason about your answer. Pounds are smaller than kilograms, so you expect your answer to be less than 205.

$$205 \text{ pounds} = \underline{\hspace{2cm}} \text{ kg}$$

2) Using the factor label method, write 205 lbs. as a fraction and use unit fractions to convert it to kilograms.

$$\frac{205 \text{ lbs}}{1} \times \frac{1 \text{ kg}}{2.2 \text{ lbs}} = \underline{\hspace{2cm}} \text{ kg}$$

3) Cancel similar units, multiply, and simplify.

$$\frac{205 \cancel{\text{ lbs}}}{1} \times \frac{1 \text{ kg}}{2.2 \cancel{\text{ lbs}}} = \frac{205 \times 1 \text{ kg}}{1 \times 2.2} = \frac{205}{2.2} \text{ kg} \approx 93.18 \text{ kg}$$

Answer: The patient's weight is 93.18 kg.

Example 15: Your work is having its annual potluck, and you are asked to bring 5 gallons of lemonade. When you go to the store, each container of lemonade mix says it will make 6 liters. How many containers do you need to buy?

Solution: State your givens and your goal.

Given: Need 5 gallons of lemonade. Lemonade mix says it will make 6 liters.

Goal: How many containers do you need to buy?

1) Convert 5 gallons to liters. Reason about your answer. Liters are smaller than gallons, so you expect your answer to be more than 5.

$$5 \text{ gallons} = \underline{\hspace{2cm}} \text{ L}$$

2) Using the factor label method, write 5 gal as a fraction and use unit fractions to convert it to liters. Cancel similar units, multiply, and simplify.

$$\frac{5 \cancel{\text{gallons}}}{1} \times \frac{3.785 \text{ L}}{1 \cancel{\text{gallon}}} = \frac{5 \times 3.785 \text{ L}}{1 \times 1} = 18.925 \text{ L}$$

3) You need 18.925 liters, and each container of mix will make 6 liters. Convert to containers.

$$\frac{18.925 \text{ L}}{1} \times \frac{1 \text{ container}}{6 \text{ L}} = \frac{18.925}{6} \text{ containers} = 3.15 \text{ containers}$$

4) Since you cannot buy a partial container, and you want to have enough lemonade, you should buy 4 containers, (3 containers would not be enough)

Answer: You need to buy 4 containers of lemonade.

Temperature Conversions

There are three commonly used systems for measuring temperature. One such system is usually used in science, and is called the **Kelvin** scale. We will focus our attention on the other two scales for measuring temperature: Fahrenheit and Celsius. The United States usually uses the **Fahrenheit** system. In this system, water freezes to ice at 32°F and boils at 212°F. Another commonly used temperature scale is the **Celsius** scale, where water freezes to ice at 0°C and boils at 100°C. You might notice that in the Fahrenheit scale there is a 180 degree difference (212°F – 32°F) between the temperature where water boils and freezes, while in the Celsius scale there is a 100 degree difference (100°C – 0°C) between the temperature where water boils and freezes. Since neither of these two temperature scales has an absolute starting point (a lowest

possible temperature) we cannot meaningfully compare temperatures in these scales using conversion factors.

Instead, we have temperature conversion formulas which allow us to convert temperatures back and forth between Fahrenheit and Celsius.

Convert from Celsius to Fahrenheit: $F = 1.8C + 32$

Convert from Fahrenheit to Celsius: $C = \frac{F - 32}{1.8}$

Example 16: Your front yard is full of weeds, so you decide to spray it with weed killer. The bottle of weed killer that you purchase says “Do not apply in temperatures below 18°C or above 32°C.” What are the corresponding temperatures in degrees Fahrenheit?

Solution: State your givens and your goal.

Given: Do not apply in temperatures below 18°C or above 32°C

Goal: What are the corresponding temperatures in degrees Fahrenheit?

1) Both temperatures are in degrees Celsius and need to be converted into degrees Fahrenheit. Use $F = 1.8C + 32$

$$\text{For } C = 18, F = 1.8(18) + 32 = 64.4^\circ F \text{ (low temperature)}$$

$$\text{For } C = 32, F = 1.8(32) + 32 = 89.6^\circ F \text{ (high temperature)}$$

Answer: The weed killer should only be applied when the temperature is between 64.4°F and 89.6°F.

Example 17: When leaving the hospital with your sick child, you are told to return immediately if her temperature exceeds 101.5°F. When you get home, you discover your thermometer will only measure temperature in degrees Celsius. At what temperature, in degrees Celsius, would you need to return your child to the hospital?

Solution: State your givens and your goal.

Given: Return immediately if her temperature exceeds 101.5°F

Goal: At what temperature, in degrees Celsius, would you need to return your child to the hospital?

1) The temperature is in degrees Fahrenheit and it needs to be converted into degrees Celsius. Use $C = \frac{F - 32}{1.8}$.

$$\text{For } F = 101.5^{\circ}\text{F}, C = \frac{101.5 - 32}{1.8} = \frac{69.5}{1.8} = 38.61^{\circ}\text{C}$$

Answer: You need to return your child to the hospital if her temperature exceeds 38.61°C .

Section 1.5 You Try Problem:

What was the temperature in degrees Celsius, if the evening news reports that the high temperature in Phoenix, Arizona today was 115°F ?

SECTION 1.6: PROBLEM SOLVING USING RATES AND DIMENSIONAL ANALYSIS

A **rate** is the ratio of two quantities. A **unit rate** is a rate with a denominator of one.

Example 18: Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate and as a unit rate, in miles per gallon.

Solution: State your givens and your goal.

Given: Your car can drive 300 miles on a tank of 15 gallons.

Goal: Express this as a rate and as a unit rate, in miles per gallon.

1) As a rate, create a fraction with the two quantities with miles in the numerator and gallons in the denominator. Include units in your answer.

$$\frac{300 \text{ miles}}{15 \text{ gallons}}$$

2) As a unit rate, divide the quantity in the numerator by the quantity in the denominator. Use the “units miles per gallon” (equivalent to miles per 1 gallon).

$$\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{300}{15} \frac{\text{miles}}{\text{gallons}} = 20 \text{ miles per gallon}$$

Notice that, had we wanted to find the unit rate in **gallons per mile**, we would have had to invert the original rate:

$$\frac{15 \text{ gallons}}{300 \text{ miles}} = \frac{15}{300} \frac{\text{gallons}}{\text{miles}} = \frac{1}{20} \text{gallons per mile}$$

Section 1.6 You Try Problem:

Find the following unit rates for 6 pounds for \$5.29. If necessary, round your answers to the nearest hundredth.

- a) In dollars per pound
- b) In pounds per dollar

Example 19: Use the following information to compare the electricity consumption per capita in China to the rate in Japan.

From the CIA¹ website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank², we can find the population of China is 1,344,130,000, and the population of Japan is 127,817,277.

Solution: State your givens and your goal.

Given: China: 4.693 trillion KWH and 1,344,130,000 people. Japan: 859.7 billion KWH and 127,817,277 people.

Goal: Computing the consumption per capita for each country and compare the results.

1) Find KWH per person for China and Japan by dividing the quantity KWH by the number of people.

$$\text{China: } \frac{4,693,000,000,000 \text{ KWH}}{1,344,130,000 \text{ people}} = 3491.48 \text{ KWH per person}$$

$$\text{Japan: } \frac{859,700,000,000 \text{ KWH}}{127,817,277 \text{ people}} = 6726 \text{ KWH per person}$$

¹ <https://www.cia.gov/library/publications/the-world-factbook/rankorder/2042rank.html>

² <http://data.worldbank.org/indicator/SP.POP.TOTL>

Answer: While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

Many problems can also be solved by multiplying a quantity by rates to change the units. This is the foundation of the Factor-Label process that we have been using already in this chapter.

Example 20: Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons? How many gallons are needed to drive 50 miles?

Solution: State your givens and your goal.

Given: Your car can drive 300 miles on a tank of 15 gallons.

Goals: a) How far can it drive on 40 gallons? b) How many gallons are needed to drive 50 miles?

a) We found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit “cancels” and we’re left with a number of miles.

$$\frac{40 \text{ gallons}}{1} \times \frac{20 \text{ miles}}{1 \text{ gallon}} = \frac{40 \times 20}{1 \times 1} \text{ miles} = 800 \text{ miles}$$

b) We also found that 300 miles on 15 gallons gives a rate of $\frac{1}{20}$ gallons per mile. If we multiply the given 50 mile quantity by this rate, the *miles* unit “cancels” and we’re left with gallons.

$$\frac{50 \text{ miles}}{1} \times \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \times 1}{1 \times 20} \text{ gallons} = 2.5 \text{ gallons}$$

Answers: The car can drive 800 miles on 40 gallons of gas. The car needs 2.5 gallons of gas to drive 50 miles.

Example 21: A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution: State your givens and your goal.

Given: 15 miles per hour.

Goal: How many feet will the bicycle cover in 20 seconds?

1) To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft = 1 mile. We might start by converting the 20 seconds into hours:

$$\frac{20 \text{ sec}}{1} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{1 \text{ hr}}{60 \text{ min}} = \frac{20 \times 1 \times 1}{1 \times 60 \times 60} \text{ hr} = \frac{20}{3600} \text{ hr} = \frac{1}{180} \text{ hr}$$

2) Now we can multiply by the 15 miles/hr

$$\frac{1 \text{ hr}}{180} \times \frac{15 \text{ mi}}{1 \text{ hr}} = \frac{15}{180} \text{ miles} = \frac{1}{12} \text{ miles}$$

3) Now we can convert to feet.

$$\frac{1 \text{ mile}}{12} \times \frac{5280 \text{ ft}}{1 \text{ mile}} = \frac{5280}{12} \text{ ft} = 440 \text{ ft}$$

4) Note we could have also done this entire calculation in one long set of products:

$$\frac{20 \text{ sec}}{1} \times \frac{1 \text{ min}}{60 \text{ sec}} \times \frac{1 \text{ hr}}{60 \text{ min}} \times \frac{15 \text{ miles}}{1 \text{ hr}} \times \frac{5280 \text{ ft}}{1 \text{ mile}} = \frac{20 \times 15 \times 5280}{60 \times 60} \text{ ft} = 440 \text{ ft}$$

Answer: The bicycle covers 440 feet in 20 seconds.

Example 22: You are walking through a hardware store and notice two sales on tubing.

- 3 yards of Tubing A costs \$5.49.
- Tubing B sells for \$1.88 for 2 feet.

Either tubing is acceptable for your project. Which tubing is less expensive?

Solution: State your givens and your goal.

Given: 3 yards of Tubing A costs \$5.49. Tubing B sells for \$1.88 for 2 feet.

Goal: Which tubing is less expensive?

1) Find the unit price per yard for each tubing. This will make it easier to compare.

Tubing A: 3 yards for \$5.49

$$\text{Unit Price per yard: } \frac{\$5.49}{3 \text{ yds}} = \frac{5.49}{3} \frac{\text{dollars}}{\text{yards}} = \$1.83 \text{ per yard}$$

Tubing B: 2 feet for \$1.88

$$\text{Unit Price per yard: } \frac{\$1.88}{2 \text{ ft}} \times \frac{3 \text{ ft}}{1 \text{ yd}} = \frac{5.64}{2} \frac{\text{dollars}}{\text{yards}} = \$2.82 \text{ per yard}$$

2) Compare prices for 1 yard of each tubing.

Tubing A: \$1.83 per yard Tubing B: \$2.82 per yard

Answer: Tubing A is less expensive than Tubing B.

Example 23: The cost of gasoline in Arizona is about \$2.05 per gallon. When you travel over the border into Mexico, gasoline costs 14.81 pesos per liter. Where is gasoline more expensive? (Note: This problem requires a currency conversion factor. Currency conversions are constantly changing, but at the time of print \$1 = 18.68 pesos.)

Solution: State your givens and your goal.

Given: Gasoline in Arizona is about \$2.05 per gallon. Gasoline in Mexico costs 14.81 pesos per liter. \$1 = 18.68 pesos

Goal: Where is gasoline more expensive?

1) We will convert the Arizona price from dollars to pesos and gallons to liters to obtain pesos per liter.

$$\frac{\$2.05}{1 \text{ gallon}} \times \frac{18.68 \text{ pesos}}{\$1} \times \frac{1 \text{ gallon}}{3.8 \text{ L}} = \frac{2.05 \times 18.68}{3.8} \frac{\text{pesos}}{\text{L}} = 10.08 \text{ pesos per liter}$$

2) Compare the pesos per liter values.

Arizona: 10.08 pesos per liter

Mexico: 14.81 pesos per liter

Answer: Gasoline is more expensive in Mexico.

Chapter 1 – Answers to You Try Problems

- 1.1 a) 13200 feet b) 930 feet or 310 yards
- 1.2 a) 4.5 pounds b) 3200 pounds
- 1.3 Answer D: The chili will fit into the 2-gallon pot only.
- 1.4 a) 42,100,000 cm 421,000 m
b) 84.2kg
- 1.5 46.1°C
- 1.6 \$0.88 per pound, 1.13 pounds per dollar

CHAPTER 1 – PRACTICE PROBLEMS

- 1) Use the Factor-Label Method to determine the number of inches in 8 feet.
- 2) Use the Factor-Label Method to determine the number of feet in 5 yards.
- 3) Use the Factor-Label Method to determine the number of inches in 6 yards.
- 4) Use the Factor-Label Method to determine the number of yards in 108 inches.
- 5) Use the Factor-Label Method to determine the number of yards in 203.1 miles. Round your answer to the nearest hundredth.
- 6) Use the Factor-Label Method to determine the number of inches in 275.2 miles.
- 7) The length of one student desk is 22 inches. If you line up 5 desks, what is the total length of the desks in feet?
- 8) You want to arrange your dorm room so that the length of your bed and the length of your desk are next to each other along the wall. Your bed is 80 inches long and your desk is 3 feet and 5 inches long. What is the total length of your bed and desk in feet?
- 9) Use the Factor-Label Method to determine the number of ounces in 4 pounds.
- 10) Use the Factor-Label Method to determine the number of pounds in 38 ounces. Round your answer to the nearest hundredth.
- 11) Use the Factor-Label Method to determine the number of tons in 413 ounces. Round your answer to the nearest thousandth.
- 12) Amy bought 5 lbs. 5 oz. of turkey cold cuts and 3 lbs. 15 oz. of ham cold cuts. How much did she buy in total? (You should convert any ounces over 15 into pounds).
- 13) Leslie had twin girls – the first baby weighed 5 lbs. 14 oz. and the second baby weighed 6 lbs. 4 oz. How much did the twin baby girls weigh together? (You should convert any ounces over 15 into pounds).
- 14) Use the Factor-Label Method to determine the number of gallons in 48 cups.
- 15) Use the Factor-Label Method to determine the number of cups in $9\frac{1}{2}$ gallons.
- 16) Use the Factor-Label Method to determine the number of fluid ounces in 9 quarts.
- 17) Betty brought one gallon of iced tea to a potluck and Denis brought 3 quarts of iced tea to the potluck. How many cups of iced tea are at the potluck?
- 18) Natasha is heating up four 18 fluid ounce cans of soup on her stovetop. What is the smallest pot she can use if she has a 2 quart, 3 quart, and 5 quart saucepan?
- 19) Use the Factor-Label Method to determine the number of centimeters in 7 meters.

- 20) Use the Factor-Label Method to determine the number of liters in 3520 milliliters.
- 21) Use the Factor-Label Method to determine the number of milliliters in 4.83 liters.
- 22) Use the Factor-Label Method to determine the number of meters in 4 millimeters.
- 23) Use the Factor-Label Method to determine the number of meters in 415 centimeters.
- 24) Use the Factor-Label Method to determine the number of kilograms in 1840 grams.
- 25) Use the Factor-Label Method to determine the number of grams in 6 kilograms.
- 26) Use the Factor-Label Method to determine the number of centimeters in 36.4 millimeters.
- 27) Use the Factor-Label Method to determine the number of milligrams in 0.0352 kilograms.
- 28) Apollo Spas services 176 hot tubs. If each hot tub needs 125 mL of muriatic acid, how many liters of acid are needed for all of the hot tubs?
- 29) A bicyclist rides 5500 meters, takes a break, and then rides an additional 4000 meters. How many kilometers did he ride?
- 30) Sandra has a small dog and two birds. Her dog weighs 3.2 kg. One bird weighs 112 grams and the other bird weighs 125 grams. How many more kg does the dog weigh when compared to the two birds?
- 31) Use the Factor-Label Method to determine the number of inches in 6 centimeters.
- 32) Use the Factor-Label Method to determine the number of centimeters in 4.5 inches.
- 33) Use the Factor-Label Method to determine the number of yards in 350 meters.
- 34) Use the Factor-Label Method to determine the number of kilometers in 7.2 miles.
- 35) Use the Factor-Label Method to determine the number of kilograms in 165 pounds.
- 36) Use the Factor-Label Method to determine the number of pounds in 0.8 kilograms.
- 37) Use the Factor-Label Method to determine the number of liters in 16 gallons.
- 38) An Olympic sprinter competes in the 400 meter dash. How many feet is the race?
- 39) Convert a temperature of $54^{\circ}F$ to degrees Celsius.
- 40) Convert a temperature of $75^{\circ}F$ to degrees Celsius.
- 41) Convert a temperature of $-12^{\circ}C$ to degrees Fahrenheit.
- 42) Convert a temperature of $37^{\circ}C$ to degrees Fahrenheit.
- 43) You arrive in Paris with \$800. How many Euros do you have to spend? (\$1 = 0.9033 euros as of November, 2016)

College Math - Chapter 1 Practice Problems

- 44) A bicyclist is traveling at 540 feet in 30 seconds. Express this as a rate and as a unit rate, in feet per second.
- 45) Your car can drive 380 miles on a tank of 22 gallons.
- a) How far can it drive on 50 gallons? Round your answer to the nearest hundredth.
 - b) How many gallons are needed to drive 650 miles? Round your answer to the nearest hundredth.
- 46) A bicycle is traveling at 16 miles per hour. How many feet will it cover in 40 seconds? Round your answer to the nearest tenth of a foot.
- 47) A car is traveling 33 miles per hour. How many feet will the car travel in 6 seconds? Round your answer to the nearest tenth of a foot.
- 48) A car is traveling at 60 kilometers per hour. How many meters does it travel in 4 seconds?
- 49) A car is driving at 100 kilometers per hour. How many meters does it travel in 2 seconds?
- 50) A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?
- 51) One 16 ounce can of vegetable soup costs \$1.34. One quart of vegetable soup costs \$3.49. Which option is less expensive?
- 52) An 8 foot length of 4 inch wide crown molding costs \$14. How much will it cost to buy 40 feet of crown molding?
- 53) A wire costs \$2 per meter. How much will 3 kilometers of wire cost?
- 54) A recipe for blueberry muffins states that it yields 12 muffins, with 270 calories per muffin. You instead decide to make mini muffins, and the recipe yields 20 mini muffins. If you eat 6 mini muffins, how many calories will you consume?
- 55) Four 3 megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?
- 56) A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?
- 57) You read online that a 15 foot by 20 foot brick patio would cost about \$2,275 to have professionally installed. Estimate the cost of having an 18 foot by 22 foot brick patio installed. Round your answer to the nearest dollar.
- 58) The store is selling lemons at 2 for \$1. Each lemon yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9 inch lemon pie requiring half a cup of lemon juice? (1 cup = 16 tablespoons)

- 59) When Ibuprofen is given for fever to children 6 months of age up to 2 years, the usual dose is 5 milligrams (mg) per kilogram (kg) of body weight. How much medicine would be a usual dose for an 18 month old weighing 28 pounds? Round your answer to the nearest milligram. (1kg = 2.2 pounds)
- 60) It is estimated that a driver takes, on average, 1.5 seconds from seeing an obstacle to reacting by applying the brake or swerving. How far will a car, traveling at 60 miles per hour, travel in feet before the driver reacts to an obstacle?
- 61) Sound travels about 750 miles per hour. If you stand in a canyon and sound a horn, you will hear an echo. Suppose it takes about 3 seconds to hear the echo. How far away is the canyon wall, in feet?
- 62) It takes a hose 2 minutes to fill a rectangular aquarium 9 inches long, 10 inches wide, and 13 inches tall. How long will it take the same hose to fill an aquarium measuring 21 inches by 27 inches by 35 inches? Round your answer to the nearest minute.
- 63) You want to put a 2 inch thick layer of topsoil for a new 20 foot by 30 foot garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order? Round your answer to the nearest tenth.
- 64) Gasoline in Paris costs 1.32 euros per liter. What is the price of gas in dollars per gallon? (\$1 = 0.9033 euros as of November, 2016)
- 65) A 0.75 liter bottle of wine costs 83 euros. What is the price of the wine in dollars per ounce? (\$1 = 0.9033 euros as of November, 2016)

CHAPTER 2 – EQUATIONS AND LINEAR BEHAVIOR

Chapter 2 Learning Objectives:

- 1) Solve one-step, two-step, and multi-step equations
- 2) Recognize a linear equation, and identify the rate of change (slope) and initial value (vertical intercept)
- 3) Solve application problems using equations
- 4) Calculate a slope and interpret it as a rate of change

Sections

Section 2.1 – Solving Equations

Section 2.2 – Applications of Solving Equations

Section 2.3 - Linear Equations

SECTION 2.1: SOLVING EQUATIONS

Definitions

- An **algebraic equation** is a mathematical sentence stating that an algebraic expression is equal to a specified value, variable, or another expression.
- The **solution** to an equation is the value, or values, that make the equation true.
- To **solve** an equation means to “undo” all the operations of the equation, leaving the variable by itself on one side. This is known as isolating the variable.

Properties of Equality

The Addition/Subtraction Property of Equality:

$$\text{If } a = b, \text{ then } a + c = b + c. \quad \text{If } a = b, \text{ then } a - c = b - c.$$

The Multiplication/Division Property of Equality:

$$\text{If } a = b, \text{ then } a \times c = b \times c \quad \text{If } a = b \text{ and } c \neq 0, \text{ then } \frac{a}{c} = \frac{b}{c}$$

Example 1: Solve for the variable in each of the following equations.

a) $10 + x = 14$

b) $b - 9 = -12$

c) $\frac{2}{3}k = -4$

Solutions: Recall, when solving an equation the goal is to isolate the variable.

a) $10 + x = 14$

Subtract 10 from both sides of the equation.

$$\begin{array}{r} 10 + x = 14 \\ -10 \quad -10 \\ \hline 0 + x = 4 \\ x = 4 \end{array}$$

Answer: $x = 4$

b) $b - 9 = -12$

Add 9 to both sides of the equation.

$$\begin{array}{r}
 b - 9 = -12 \\
 +9 \quad +9 \\
 \hline
 b + 0 = -3 \\
 b = -3
 \end{array}$$

Answer: $b = -3$

c) $\frac{2}{3}k = -4$

Divide both sides by $\frac{2}{3}$ or equivalently, multiply by $\frac{3}{2}$

$$\begin{array}{r}
 \frac{2}{3}k = -4 \\
 \left(\frac{3}{2}\right) \times \frac{2}{3}k = \left(\frac{3}{2}\right) \times -4 \\
 1 \times k = \frac{-12}{2} \\
 k = -6
 \end{array}$$

Answer: $k = -6$ **Section 2.1A You Try Problems:**

Solve for the variable in each of the following equations.

a) $3 + c = -10$

b) $\frac{4}{5}r = 2$

Each of the problems in Example 1 are examples of solving one-step equations. These are equations that require only one step to solve. In many cases, two or even multiple steps are required to isolate the variable.

Steps for Solving a Multi-Step Equation

- 1) Simplify each side of the equation. Remove parenthesis if necessary. Combine like terms.
- 2) Add or subtract terms on each side of the equation so that all terms containing the variable are on one side and all constant terms are on the other side.
- 3) Simplify each side of the equation by combining like terms.
- 4) Apply the Multiplication/Division Property of Equality to isolate the variable.

Note: You can check your answer by substituting the solution into the original equation.

Example 2: Solve for the variable in each of the following equations.

$$a) \ 5 = 2x + 7 \quad b) \ 480 = P + 2.2P \quad c) \ 2000 = 500(1 + 0.04t) \quad d) \ \frac{x-4}{2.2} = 1.85$$

Solutions:

- a) We begin by subtracting 7 so that the constant term is on one side of the equation, and the variable term is on the other side. Then divide both sides by 2

$$\begin{array}{rcl} 5 & = & 2x + 7 \\ -7 & & -7 \quad \text{subtract 7} \\ \hline -2 & = & 2x + 0 \\ \frac{-2}{2} & = & \frac{2x}{2} \quad \text{divide by 2} \\ -1 & = & x \end{array}$$

Answer: $x = -1$

- b) Start by combining like terms on the right side. Then divide both sides by 3.2.

$$\begin{array}{rcl} 480 & = & P + 2.2P \quad \text{Add : } 1P + 2.2P = (1 + 2.2)P = 3.2P \\ \frac{480}{3.2} & = & \frac{3.2P}{3.2} \quad \text{Divide by 3.2} \\ 150 & = & P \end{array}$$

Answer: $P = 150$

- c) Divide both sides by 500. Subtract 1 from both sides. Divide both sides by 0.04

$$\begin{array}{rcl} 2000 & = & 500(1 + 0.04t) \\ \frac{2000}{500} & = & \frac{500(1 + 0.04t)}{500} \quad \text{Divide by 500} \\ 4 & = & 1 + 0.04t \quad \text{Subtract 1} \\ -1 & & -1 \\ \hline 3 & = & 0.04t \\ \frac{3}{0.04} & = & \frac{0.04t}{0.04} \quad \text{Divide by 0.04} \\ 75 & = & t \end{array}$$

Answer: $t = 75$

d) Multiply both sides by 2.2. Add 4 to both sides.

$$\begin{aligned}\frac{x-4}{2.2} &= 1.85 \\ 2.2 \times \frac{x-4}{2.2} &= 2.2 \times 1.85 && \text{Multiply both sides by 2.2} \\ x-4 &= 4.07 \\ \begin{array}{r} +4 \quad +4 \\ \hline \end{array} &&& \text{Add 4 to both sides} \\ x &= 8.07\end{aligned}$$

Answer: $x = 8.07$

Section 2.1B You Try Problems:

Solve for the variable in each of the following equations.

a) $-9 = 4x + 3$ b) $400 = 25(1 + 0.03t)$

SECTION 2.2: APPLICATIONS OF SOLVING EQUATIONS

When solving application problems, first determine the Givens and the Goal, then form a Strategy, Solve, and Check. Write your answer in a complete sentence.

Example 3: The maximum heart rate is the highest heart rate achieved during maximal exercise. In general, you gain the most benefits and lessen the risks when you exercise within your *target* heart rate zone. Usually this is when your exercise heart rate (pulse) is about 70% percent of your maximum heart rate. The formula $T = 0.7(220 - a)$ gives the target heart rate, T , in beats per minute, for a person who is a years of age. Determine the age of a person whose target heart rate is 133 beats per minute.

Solution: State your givens and your goal.

Given: T = target heart rate, a = age in years, Formula: $T = 0.7(220 - a)$

Goal: Determine the age of a person whose target heart rate is 133 beats per minute or equivalently, find the age a , for $T = 133$.

Strategy: We want to find the age of a person who has a target heart rate of 133. We know that $T = 133$, and we need to solve for a . Plug in the value of $T = 133$ to the given equation, and solve for a .

$T = 0.7(220 - a)$	Substitute 133 in the equation for T
$133 = 0.7(220 - a)$	Divide both sides by 0.7
$190 = 220 - a$	Subtract 220 from both sides
$-30 = -a$	Divide both sides by -1
$30 = a$	

Answer: $a = 30$ years old

Check: Does $0.7(220 - 30)$ equal 133?

$$0.7(190) = 133 \quad \text{Yes it checks.}$$

Final Result as a Complete Sentence: A person with a target heart rate of 133 beats per minute is 30 years old.

Section 2.2 You Try Problem:

For this problem, identify the Givens and the Goal. Form a strategy, solve, check, and write your answer in a complete sentence. Show all steps.

The cost of tuition at a local community college is given by the equation $C = 76n$ where C represents the total cost of tuition and n represents the number of credits taken. If you have \$800 dollars to spend on tuition, how many credits can you take?

SECTION 2.3: LINEAR EQUATIONS

All of the equations in the previous section had 2 variables. In those examples, we were given the value of one variable, and asked to solve for the other.

There are many situations in the “real world” where two variables tend to change together. When paying your power bill, you might pay a base fee plus a specific dollar amount for each kilowatt hour that you use. When buying tickets to a concert, you might pay a cost per ticket that you buy, plus a one-time convenience fee. While running on the treadmill at the gym, you might notice that you run a certain number of miles for each minute that you run.

Each of the cases above are examples where two variables (quantities) are changing together. Specifically, in each of these cases there is a **CONSTANT** rate of change relating the two variables. That **constant rate of change** is what makes the relationship between the two quantities **linear**.

<u>Situation</u>	<u>Rate of Change in Words</u>	<u>Sample Rate of Change</u>
1) Power Bill	Cost per kilowatt hour	15.4 cents per kilowatt hour
2) Concert Tickets	Cost per ticket	\$155 per ticket
3) Treadmill	Miles run per minute	0.1 miles per minute

In mathematics, we typically use variables x and y to model two variables that change together. Here x is called the input or independent variable (and is plotted on the horizontal axis), and y is called the output or dependent variable (and is plotted on the vertical axis). Let's look at how we would define x and y for the situations modeled previously:

<u>Rate of Change in Words</u>	<u>Independent Variable (x)</u>	<u>Dependent Variable (y)</u>
1) Cost per kilowatt hour	Kilowatt hours of electricity used	Cost
2) Cost per ticket	Number of tickets	Cost
3) Miles run per minute	Number of minutes run	Miles run

Notice that in each of the cases above, we assign the independent and dependent variables by looking at how the dependent variable changes **per** unit change in the independent variable: "y per x".

Summary of Linear Equations

A linear equation is an equation whose graph produces a line. It is used to model a linear relationship between two variables: x and y .

Linear equations can be written in several different forms, the most common of which is called slope-intercept form:

$$y = mx + b \quad \text{or} \quad y = b + mx$$

(The two equations above are equivalent)

where b is the initial or starting value (the value of y when $x = 0$, also called the vertical intercept), and m is the constant rate of change (also called the slope).

Many people like to write linear equations in the form $y = b + mx$ because it corresponds to the way we tend to speak: "The value of the dependent variable starts at b and changes at a rate of m ."

For this reason alone we will use the form $y = b + mx$ for many of the examples, but remember $y = mx + b$ and $y = b + mx$ are equivalent and can be written correctly either way.

Here, m is the constant rate of change of the equation (also called the **slope**). The slope determines if the line is increasing from left to right, or decreasing from left to right.

Slope	Behavior	Example Graph
$m > 0$	Increasing	
$m < 0$	Decreasing	
$m = 0$	Horizontal	
m is undefined	Vertical	

Example 4: In the table below are several linear equations, and their corresponding slopes, behavior, and vertical intercept (initial value).

Equation	Slope	Behavior	Vertical Intercept
$y = 3x + 5$	3	Increasing (since the slope is positive)	(0, 5)
$y = 8 - x$	-1	Decreasing (since the slope is negative)	(0, 8)
$y = 2x$	2	Increasing (since the slope is positive)	(0, 0)
$y = -8$	0	Horizontal (since the slope is 0)	(0, -8)

Up to this point we have used x and y to represent the two variables we are modeling, but any two variables can be used. Frequently, it is easier to choose variables that make sense given the context of the situation.

Example 5: Marcus currently owns 200 songs in his iTunes collection. Every day, he adds 3 new songs. Write a formula for the number of songs, N , in his iTunes collection after d days. How many songs will Marcus own in one year?

Solution:

Givens: Currently owns 200 songs, every day, he adds 3 new songs

Goals: a) Write a formula for the number of songs, N , in his iTunes collection after d days.
b) How many songs will Marcus own in one year?

a) The initial value (vertical intercept) for this equation is 200, since he currently owns 200 songs. We know this can be represented as the ordered pair (0, 200). The number of songs increases by 3 songs per day, so the rate of change (or slope) is 3 songs per day. With this information, we can write the formula:

$$N = 200 + 3d$$

b) With this formula we can predict how many songs he will have in 1 year (365 days):

$$N = 200 + 3(365)$$

$$N = 1295$$

Answer: The formula is $N = 200 + 3d$. Marcus will have 1295 songs in 365 days (or 1 year).

Section 2.3A You Try Problem:

You have just bought a new Sony 55" 3D television set for \$2300. The TV's value decreases at a rate of \$250 per year. Construct a linear equation to represent this situation. Clearly indicate what your variables represent.

Calculating Rate of Change

Given any two points, (x_1, y_1) and (x_2, y_2) , if we wish to find a linear equation that contains both points we can calculate the rate of change, m :

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Recall, the rate of change of a linear equation is also called the slope of the line.

Example 6: The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution: First determine the givens and the goals

Givens: The population of a city increased from 23,400 to 27,800 between 2002 and 2006.

Goals: Find the rate of change of the population during this time span.

1) The rate of change will relate the change in population to the change in time. **When time is one of your variables, use it as the input variable.** In this example, we have ordered pairs where the input is time (in years) and the output is population. We can write two ordered pairs: (2002, 23400) and (2006, 27800). Recall that the rate of change, or slope is:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

2) In this example, we have

$$m = \frac{27800 - 23400}{2006 - 2002}$$

3) The population increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$m = \frac{4400 \text{ people}}{4 \text{ years}} = 1100 \text{ people per year}$$

(Note: You should recognize the slope as a unit rate from chapter 2!)

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Answer: The rate of change of is 1100 people per year.

Section 2.3B You Try Problem:

In the year 1998, the surface elevation of Lake Powell was 3,843 feet above sea level. In the year 2001, the surface elevation of Lake Powell was 3,609 feet above sea level. Determine the rate of change in this situation.

We can now find the rate of change given two input-output pairs, and can write an equation for a line once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the equation, then we will have to solve for it.

Example 7: Working as an insurance salesperson, Dimetre earns a base salary and a commission on each new policy, so Dimetre's weekly income, I , depends on the number of new policies, n , he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for I in terms of n , and interpret the meaning of the components of the equation.

Solution: First determine the givens and the goals

Givens: Dimetre's weekly income, I , depends on the number of new policies, n , he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920.

Goal: Find an equation for I in terms of n , and interpret the meaning of the components of the equation.

1) From the given information we can write two input-output pairs in the form (n, I) . Those two ordered pairs are: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

2) Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is $\frac{\$80}{1\text{policy}}$. So Dimetre earns a commission of \$80 for each policy sold during the week.

3) Next we can then solve for the initial value using our $m = 80$ and the linear form $I = b + 80n$.

$I = b + 80n$	Original formula with $m = 80$
$760 = b + 80 \times 3$	Replacing (n, I) with the pair $(3, 760)$
$760 = b + 240$	Multiply 80 and 3
$760 - 240 = b$	Subtract 240 from each side to solve for b
$520 = b$	

This value ($b = 520$) is the starting value of Dimetre's income. This is Dimetre's income when $n = 0$, which means no new policies are sold. We can interpret this as Dimetre's base salary for the week, which does not depend upon the number of policies sold.

4) Write the final equation based on m and b : $I = 520 + 80n$

Answer: Dimetre's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold during the week.

Section 2.3C You Try Problem:

The balance in your college payment account, C , depends on the number of semesters, s , you attend. Interpret the equation $C = 20000 - 4000s$ in words.

Problem Solving Strategy

- 1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
- 2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the model, like slope and initial value. Also identify what you are trying to find, identify, solve, or interpret.
- 3) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the equation being used to model the problem.
- 4) Solve or evaluate using the formula you found for the desired quantities.
- 5) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
- 6) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

This strategy will help you work through the following examples.

Example 8: Arielle saved up \$3500 for her summer visit to Seattle. She anticipates spending \$400 each week on rent, food, and fun. How long can she afford to stay in Seattle?

Solution: In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

Output: M , money remaining, in dollars

Input: t , time, in weeks

Reading the problem, we identify two important values. The first, \$3500, is the initial value for M . The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the question, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M = 3500 - 400t$, where M is the amount of money remaining after t weeks.

To find out how long she can stay, we need to find out how long it will take for her to use up all of the money she has saved. So we are looking for the number of weeks, t , when $M = 0$. Set the output to zero, and solve for the input:

$$0 = 3500 - 400t$$

$$t = \frac{3500}{400} = 8.75$$

Interpreting this, we could say: **Arielle will have no money left after 8.75 weeks.**

Example 9: Jamaal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of \$20, then 59 cents per mile. The second, Budget, charges an up-front fee of \$16, then 63 cents per mile³. When will U-Haul be the better choice for Jamal?

Solution: The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two equations:

Input: m , miles driven

Outputs: U : cost, in dollars, for renting from U-Haul

B : cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 a mile for U-Haul, and \$0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $U < B$.

Using the rates of change and initial charges, we can write the equations:

$$U = 20 + 0.59m \qquad B = 16 + 0.63m$$

³ Rates retrieved Aug 2, 2010 from <http://www.budgettruck.com> and <http://www.uhaul.com/>

These graphs are sketched to the right, with the graph of U drawn with a dashed line.

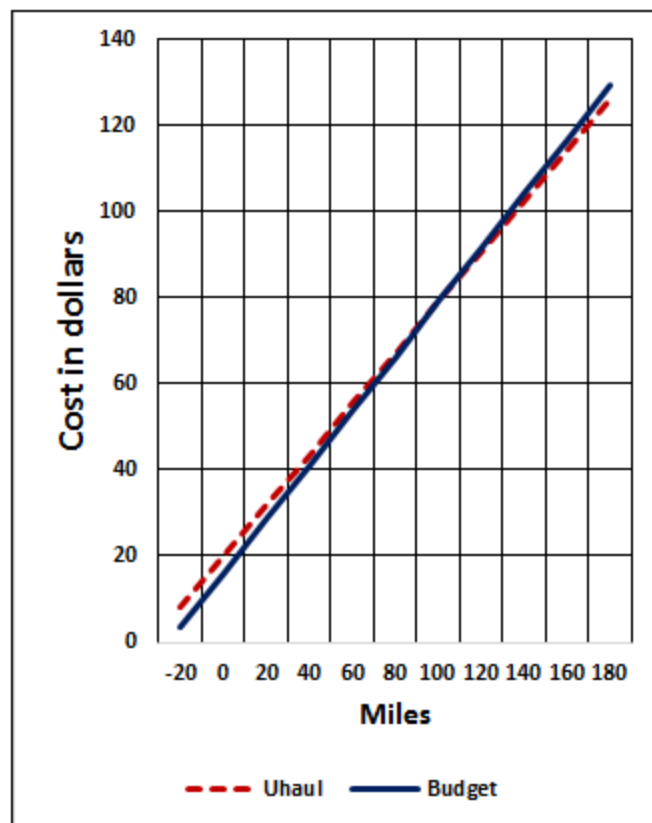
To find the intersection, we set the equations equal and solve: $U = B$.

$$20 + 0.59m = 16 + 0.63m$$

$$-0.04m = -4$$

$$m = 100$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that U is growing at a slower rate, we can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.



Example 10: A town's population has been growing linearly. In 2004, the population was 6,200. By 2009, the population had grown to 8,100. If this trend continues,

- Predict the population in 2013
- When will the population reach 15,000?

Solution: The two changing quantities are the population and time.

Input: t , years since 2004

Output: P , the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$ (since 2004 is 0 years since 2004). This produces the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the equation. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = \frac{380 \text{ people}}{1 \text{ year}} = 380 \text{ people per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation: $P = 6200 + 380t$

To predict the population in 2013, we evaluate our equation at $t = 9$

$$P = 6200 + 380(9) = 9620$$

Answer a): If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P = 15000$ and solve for t .

$$P = 6200 + 380t$$

$$15,000 = 6200 + 380t \quad (\text{replace } P \text{ with } 15,000)$$

$$8800 = 380t \quad (\text{subtract } 6200)$$

$$t \approx 23.158 \quad (\text{divide by } 380)$$

Answer b): Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Section 2.3D You Try Problems:

In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.

- How much did the population grow between the year 2004 and 2008?
- How long did it take the population to grow from 1001 students to 1697 students?
- What is the average population growth per year?
- Find an equation for the population, P , of the school t years after 2004.
- Using your equation, predict the population of the school in 2011.

Chapter 2 – Answers to You Try Problems

2.1A: a) $c = -13$ b) $r = \frac{5}{2}$

2.1B: a) $x = -3$ b) $t = 500$

2.2: If you have \$800 to spend, you can take 10 credits.

2.3A: $V = 2300 - 250n$, where V gives the value (in dollars) of the TV after n years.

2.3B: The surface elevation of Lake Powell is decreasing at a rate of 78 feet per year.

2.3C: Your College account starts with \$20,000 in it and you withdraw \$4,000 each semester (or your account contains \$20,000 and decreases by \$4000 each semester.) You can pay for 5 semesters before the money in this account is gone.

2.3D: a) 696 students b) 4 years c) 174 students per year

d) $P = 174t + 1001$ e) 2219 students

CHAPTER 2 – PRACTICE PROBLEMS

For problems 1–9, solve for the variable in each of the following equations. Reduce and simplify.

1) $3x = 9$

2) $\frac{7}{8}x = -2$

3) $8k - 2 = 30$

4) $5 - k = 3$

5) $-\frac{1}{2}x - 4 = 8$

6) $\frac{2}{3}x + 3 = 15$

7) $\frac{x-15}{13.6} = -2.40$

8) $7000 = 3500(1 + 0.04t)$

9) $5500 = 1000(1 + 0.025t)$

For problems 10 – 12 below, *underline* the Givens and *circle* the Goal of the problem. Form a Strategy, Solve, and Check. Show all work, and write your answers in complete sentences.

10) John is a door to door vacuum salesman. His weekly salary, S , is \$200 plus \$50 for each vacuum he sells. This can be written as $S = 200 + 50v$, where v is the number of vacuums sold. If John earns \$1000 for a week's work, how many vacuums did he sell?

11) Paul is planning to sell bottled water at the local Lollapalooza. He buys 2 crates of water (2000 bottles) for \$360 and plans on selling the bottles for \$1.50 each. Paul's profit, P in dollars, from selling b bottles of water is given by the formula $P = 1.5b - 360$. How many bottles does Paul need to sell in order to break even? (hint: break-even point is when profit = 0)

12) The formula for the area, A , of a triangle with base b and height h is $A = \frac{1}{2}bh$. Determine the height of a triangle with a base of 18 inches and area 84.6 square inches. Round your answer to the nearest tenth, and include appropriate units in your answer.

- 13) Identify the rate of change and initial value for the equation $y = 3x + 7$
- 14) Identify the rate of change and initial value for the equation $y = -9 - 10x$
- 15) Identify the rate of change and initial value for the equation $y = -5x$
- 16) Identify the rate of change and initial value for the equation $y = 1$
- 17) Calculate the rate of change of the linear equation that contains the points (1, 8) and (4, 17).
- 18) Calculate the rate of change of the linear equation that contains the points (-15, 30) and (7, -14).
- 19) Calculate the rate of change of the linear equation that contains the points (3,2) and (10,2)
- 20) Determine the rate of change and initial value of the linear equation that generates the following table of values.

x	-4	-3	-2	-1	0	1	2	3	4
y	-70	-49	-28	-7	14	35	56	77	98

- 21) Determine the rate of change and initial value of the linear equation that generates the following table of values.

x	-4	-3	-2	-1	0	1	2	3	4
y	102	75	48	21	-6	-33	-60	-87	-114

- 22) Determine the rate of change and initial value of the linear equation that generates the following table of values.

x	-4	-2	0	2	4
y	-90	-14	62	138	214

- 23) The equation $V = -5.4t + 34.2$ gives the value (in thousands of dollars) of an investment after t years. Interpret the rate of change in this situation.
- 24) The equation $V = 2.5x + 25.4$ gives the value (in thousands of dollars) of an investment after x months. Interpret the rate of change in this situation.
- 25) Paul is planning to sell bottled water at the local carnival. Paul's profit (in dollars) from selling b bottles of water is given by: $P = 1.4b - 338$. Interpret the rate of change in this situation.

College Math – Chapter 2 Practice Problems

- 26) When a new charter school opened in 1996, there were 440 students enrolled. Write a formula for the equation N , representing the number of students attending this charter school t years after 1996, assuming that the student population:
- Increased by 44 students per year
 - Decreased by 32 students per year
 - Remained constant (did not change)
- 27) A town's population has been growing linearly. In 2003, the population was 59,000 and the population has been growing by 1,700 people each year.
- Write a formula for the population x years after 2003.
 - What will the town's population be in 2007?
 - In what year will the population be 77,700 people?
- 28) Last year, Pinwheel Industries introduced a new toy. It cost \$1,300 to develop the toy and \$30 to manufacture each toy.
- Write a formula for the total cost to produce n of these toys.
 - What is the total cost to produce 4,400 toys?
 - How many total toys can be produced with \$124,300?
- 29) In the year 1999, the surface elevation of Lake Powell was 3848 feet above sea level. In the year 2007, the surface elevation of Lake Powell was 3,227.2 feet above sea level. Find the rate of change of the surface elevation during this time span.
- 30) In the year 1987, an investment was worth \$29,800. In the year 1997, this investment was worth \$43,800.
- Find the rate of change of the investment during this time span.
 - Is the investment increasing or decreasing?
- 31) In the year 1982, an investment was worth \$26,200. In the year 1990, this investment was worth \$39,000.
- Find the rate of change of the investment during this time span.
 - Is the investment increasing or decreasing?
- 32) In 1995, the cost of tuition at a large Midwestern university was \$170 per credit hour. In 2000, tuition had risen to \$235 per credit hour.
- Find the rate of change of the cost of tuition during this time span.
 - Write a formula for the cost of tuition in terms of the number of years since 1990.
 - What will the tuition be in 2007?
 - What year will the tuition be \$417 per credit hour?

- 33) Robert has \$100 in his school lunch account. If the school charges \$3.50 per lunch, how many lunches can he purchase? How much money will remain in his account?
- 34) You are interested in starting a gym membership and have narrowed your selection to two gyms. The first, Phoenix Fitness, charges a one-time membership fee of \$99 and charges \$35 per month. The second, Slim Gym, charges a one-time membership fee of \$10 and charges \$40 per month.
- a) Which gym is the more affordable choice if you are going to commit to a 1–year membership?
 - b) Which gym is the more affordable choice if you are going to commit to a 2–year membership?

CHAPTER 3 – PERCENTAGES

Chapter 3 Learning Objectives:

- 1) Convert between fractions, decimals, and percentages
- 2) Solve for part, whole, or percent given any two of the three values
- 3) Calculate and interpret both absolute and relative change
- 4) Find percent increase and percent decrease
- 5) Recognize an exponential equation and identify both the growth factor and initial value
- 6) Determine the growth or decay rate from an exponential equation
- 7) Model real world problems using exponential equations

Sections

Section 3.1 – Percentage Basics

Section 3.2 – Applications Involving Percentages

Section 3.3 – Absolute and Relative Change

Section 3.4 – Percent Increase and Percent Decrease

Section 3.5 – Exponential Equations

SECTION 3.1: PERCENTAGE BASICS

Percent literally means “per 100,” or “parts per hundred.” When we write 40%, this is equivalent to the fraction $\frac{40}{100}$ or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since $\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$.

Example 1: Write each as a percent:

a) $\frac{1}{4}$

b) 0.02

c) 2.35

Solutions:

a) Rewrite the fraction as a decimal and then multiply by 100

$$\frac{1}{4} = 0.25 = 0.25 \times 100\% = 25\%$$

Answer: 25%

b) Multiply by 100%

$$0.02 = 0.02 \times 100\% = 2\%$$

Answer: 2%

c) Multiply by 100%

$$2.35 = 2.35 \times 100\% = 235\%$$

Answer: 235%

Section 3.1 You Try Problems:

Fill in the blank spaces for the corresponding fraction, decimal, or percent for the given value.

	<u>Fraction</u>	<u>Decimal</u>	<u>Percent</u>
a)	$\frac{3}{5}$	_____	_____
b)	_____	0.02	_____
c)	_____	_____	72%
d)	_____	0.025	_____
e)	$4\frac{1}{2}$	_____	_____

SECTION 3.2: APPLICATIONS INVOLVING PERCENTAGES**Percents**

If we have a part that is some percent of a whole, then

$$\text{percent} = \frac{\text{part}}{\text{whole}}, \text{ or equivalently, } \text{part} = \text{percent} \times \text{whole}$$

To do the calculations, we write the percent as a decimal.

Example 2: 243 people out of 400 state that they like dogs. What percent is this?

Solution: State your givens and your goal.

Given: 243 people out of 400 state that they like dogs

Goal: Find the percent of people who like dogs.

Strategy: Write the ratio of people who like dogs to the total number of people. Divide to write the ratio as a decimal. Multiply by 100% to convert the decimal to a percent.

$$\frac{243}{400} = 0.6075 = 0.6075 \times 100\% = 60.75\%$$

Answer: 60.75% of people state that they like dogs.

Section 3.2A You Try:

To win the election as president of the United States of America, a person must obtain 270 out of 538 possible votes from the Electoral College. What percentage of the overall electoral votes is this? Round your answer to the nearest tenth of a percent.

Example 3: The sales tax in a town is 9.4%. How much tax will you pay on a \$140 purchase?

Solution: State your givens and your goal.

Given: Sales tax is 9.4%. Purchase is \$140

Goal: Find the corresponding tax in dollars.

Strategy: The tax rate is given as a percentage. The purchase price is the whole amount we are finding a percentage “of”, so the tax dollars is the part. Use $\text{part} = \text{percent} \times \text{whole}$

$$\text{tax} = \text{tax rate} \times \text{purchase amount} = 9.4\% \times \$140 = 0.094 \times \$140 = \$13.16$$

Answer: The tax amount is \$13.16.

Section 3.2B You Try:

In a recent poll, 28% of the 750 individuals polled indicated that they would vote purely Democratic in the next election. How many of the individuals would vote a straight Democratic ticket?

Example 4: A lender requires a minimum down payment of 12% of the value of the home. You have \$22,020 cash available to use as a down payment toward a home. Determine the maximum home value that you can finance.

Solution: State your givens and your goal.

Given: minimum down payment of 12% of the value of the home. You have \$22,020 cash.

Goal: Determine the maximum home value that you can finance.

Strategy: To compute the maximum home value, we need to understand what the \$22,020 represents. This is the down payment; we need to find the value of the home. In this case, the down payment is the “part” and the home value is the “whole.”

Recall that $\text{part} = \text{percent} \times \text{whole}$, so equivalently, $\text{whole} = \frac{\text{part}}{\text{percent}}$. (Remember to write 12% as the decimal 0.12 to ease our computations.)

$$\text{Whole} = \text{Home value} = \frac{\$22,020}{0.12} = \$183,500$$

Answer: The maximum home value that you can finance is \$183,500.

Section 3.2C You Try:

One banana contains about 425mg of potassium. That is about 13% of the recommended daily amount of potassium. How much potassium should be consumed daily?

When working with percentages, it is very important to understand the quantities being compared. Consider the examples below.

Example 5: In the 2004 vice-presidential debates, Edwards's claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Reasoning: Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percentages are talking about different things, so one does not necessarily contradict the other. Edward's claim was a percent of *coalition forces*, while Cheney's claim was a percent with *both coalition and Iraqi security forces*. It turns out both statistics are in fact fairly accurate.

Example 6: Over the basketball season, Isaac scores on 40% of 2-point field goal attempts, and on 30% of 3-point of field goal attempts. Find Isaac's overall field goal percentage.

Reasoning: It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more 2-point attempts than 3-point attempts. We don't actually have enough information to answer the question. Suppose Isaac attempted 200 2-point field goals and 100 3-point field goals. Then he made $200(0.40) = 80$ 2-point shots and $100(0.30) = 30$ 3-point shots. Overall, Isaac made 110 shots out of 300, for an $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

SECTION 3.3: ABSOLUTE AND RELATIVE CHANGE

Oftentimes, percentages are used to compare how a quantity has changed over time. When making this type of comparison, we use absolute change and relative change.

Absolute and Relative Change

Given two quantities,

Absolute change = ending quantity – starting quantity

Absolute change has the same units as the original quantity.

$$\text{Relative change} = \frac{\text{absolute change}}{\text{starting quantity}} \quad (\text{multiply by 100 to write as a \%})$$

Relative change gives a **percent** change.

Example 7: The value of a car dropped from \$7,400 to \$6,800 over the last year. What percent decrease is this?

Solution: State your givens and your goal.

Given: The value of a car dropped from \$7,400 to \$6,800 over the last year

Goal: What percent decrease is this?

Strategy: To compute the percent change, we first need to find the dollar value change:

$$\$6,800 - \$7,400 = -\$600.$$

The **absolute change** is -600 .

Since we are computing the decrease relative to the starting value, we compute this percent out of \$7,400:

$$\frac{-600}{7,400} = -0.081 = -8.1\%$$

The value of -8.1% is called a **relative change**.

The fact that it is negative indicates that the value of the car **decreased** over the last year.

Answer: The percent decrease is 8.1%.

Example 8: There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

Solution:

Givens: 75 QFC supermarkets, 215 Albertsons

Goals: Compare the size of the two companies.

Strategy: When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is $215 - 75 = 140$. From this, we could say “Albertsons has 140 more stores than QFC.” However, if you wrote this in an article or paper, that number does not mean much. The relative difference may be more meaningful.

There are two different relative changes we could calculate, depending on which store we use as the base:

1) Using QFC as the base, $\frac{140}{75} = 1.867 = 186.7\%$. This tells us Albertsons is 186.7% larger than QFC.

2) Using Albertsons as the base, $\frac{140}{215} = 0.651 = 65.1\%$. This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC:

3) $\frac{215}{75} = 2.867 = 286.7\%$, which tells us Albertsons is 2.867 times the size of QFC.

Likewise, we could calculate the size of QFC relative to Albertsons:

4) $\frac{75}{215} = 0.349 = 34.9\%$ which tells us that QFC is 34.9% of the size of Albertsons.

Answer: There are various comparisons as stated above.

Example 9: Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution:

Givens: a stock drops in value by 60% one week, increases in value the next week by 75%

Goals: Is the value higher or lower than where it started?

Strategy: To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%:

$$\$100 - \$100(0.60) = \$100 - \$60 = \$40.$$

In the next week, notice the base of the percent has changed to the new value, \$40. Computing the 75% increase:

$$\$40 + \$40(0.75) = \$40 + \$30 = \$70.$$

In the end, the stock is still \$30 lower, which is $\frac{\$30}{\$100} = 30\%$ lower, than it started.

Answer: The stock is lower than when it started.

Section 3.3 You Try Problem:

The U.S. federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006⁴. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Example 10: A Seattle Times article on high school graduation rates reported “The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as “dropout factories” – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half.”

- a) Is the “decrease by 17” number a useful comparison?
- b) Considering the last sentence, can we conclude that the number of “dropout factories” was originally 34?

Solution:

- a) This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of “dropout factories” dropped from 20 to 3, that’d be a very significant change, but if the number dropped from 217 to 200, that’d be less of an improvement.

⁴ <http://www.whitehouse.gov/sites/default/files/omb/budget/fy2013/assets/hist07z1.xls>

b) The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of “dropout factories” was probably previously around 34. However, it’s possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 11: A politician’s support increases from 40% of voters to 50% of voters. Describe the change.

Solution: When talking about a change of quantities that are already measured in percentages, we have to be careful in how we describe the change.

We could describe this using an absolute change $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we’d say the politician’s support has increased by 25%.

Answer: There is an absolute increase of 10% and a relative increase of 25%.

SECTION 3.4: PERCENT INCREASE AND PERCENT DECREASE

In the previous section, we calculated the relative, or percent, change. Now, we will use the idea of percent change to determine the future value of the quantity.

Example 12: In the news, you hear “tuition is expected to increase by 7% next year.” If tuition this year was \$1,200 per quarter, what will it be next year?

Solution: There are a couple of ways to approach this problem:

- 1) The tuition next year will be the current tuition plus an additional 7%, so it will be $100\% + 7\% = 107\%$ of this year’s tuition. Calculate 107% of \$1,200:

$$\$1,200(1.07) = \$1,284.$$

- 2) Alternatively, we could have first calculated 7% of \$1,200: $\$1,200(0.07) = \84 .

Notice this is NOT the expected tuition for next year (we could only wish). Instead, this is the expected increase, so to calculate the expected tuition, we’ll need to add this change to the previous year’s tuition:

$$\$1200 + \$84 = \$1,284.$$

Answer: Tuition will be \$1284 next year.

Example 13: A TV originally priced at \$869 is on sale for 35% off. Determine the sale price of the TV.

Solution: There are a couple of ways to approach this problem:

- 1) After the discount, you will wind up paying $100\% - 35\% = 65\%$ of the original price of the item. Calculate 65% of the original price to find the sale price:

$$\$869(0.65) = \$564.85$$

- 2) Alternatively, we could have first calculated 35% of \$869: $\$869(0.35) = \304.15 .

Notice this is the amount of the discount, NOT the sale price! So, to calculate the sale price, we will need to subtract discount from the original price of the item:

$$\$869 - \$304.15 = \$564.85.$$

Answer: The sale price of the TV is \$564.85.

Section 3.4 You Try Problem:

- a) The bill for dinner (after tax) was \$85.20. You decide to leave a 15% tip. Calculate the total amount paid.
- b) A clothing store is having a 15% off sale. Determine the sale price for an item that was originally priced at \$74.

Example 14: You purchased a new phone in Scottsdale, which has 7.95% sales tax. The total cost (including tax) for your new phone was \$505.40. What was the pre-tax price?

Solution: To compute the pre-tax price, we need to understand what the \$505.40 represents. Since \$505.40 is the total cost including tax, we know:

$$\text{Cost of phone} + \text{tax} = \$505.40$$

We know the dollar amount of tax is 7.95% of the cost of the phone, or $0.0795(\text{cost of phone})$. If we call the cost of the phone C , we can write the equation:

$$C + 0.0795C = 505.40$$

Combine like terms

$$1.0795C = 505.40$$

Divide to solve for C

$$C = \$468.18$$

Answer: The pre-tax cost of the phone was \$468.18.

SECTION 3.5: EXPONENTIAL EQUATIONS

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year⁵. We might ask if we can find a formula to model the population, P , in terms of time, t , in years after 2008, if the population continues to grow at this rate.

In **linear growth**, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation $y = 3x + 4$, the slope tells us the output increases by three each time the input increases by one.

This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change.

To see the significance of this difference consider these two companies:

- Company A has 100 stores, and expands by opening 50 new stores a year
- Company B has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

Year	Stores, company A	Stores, company B	Comparison
0	100	100	Starting with 100 each
1	$100 + 50 = 150$	$100 + 50\% \text{ of } 100$ $100 + 0.50(100) = 150$	They both grow by 50 stores in the first year.
2	$150 + 50 = 200$	$150 + 50\% \text{ of } 150$ $150 + 0.50(150) = 225$	Store A grows by 50, Store B grows by 75
3	$200 + 50 = 250$	$225 + 50\% \text{ of } 225$ $225 + 0.50(225) = 337.5$	Store A grows by 50, Store B grows by 112.5

Notice that with the percent growth, each year the company grows by 50% of the current year's total, so as the company grows larger, **the number of stores added** in a year grows as well. To try to simplify the calculations, notice that after 1 year the number of stores for company B was: $100 + 0.50(100)$ or equivalently by factoring $100(1 + 0.50) = 150$. We can think of this as "the new number of stores is the original 100% plus another 50%".

⁵ World Bank, World Development Indicators, as reported on <http://www.google.com/publicdata>, retrieved August 20, 2010

After 2 years, the number of stores was:

$$150 + 0.50(150) \text{ or equivalently by factoring}$$

$150(1 + 0.50)$ now recall the 150 came from $100(1 + 0.50)$. Substituting that,

$$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$$

After 3 years, the number of stores was:

$$225 + 0.50(225) \quad \text{or equivalently by factoring}$$

$225(1 + 0.50)$ now recall the 225 came from $100(1 + 0.50)^2$. Substituting that,

$$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.50$$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of $(1 + 0.50)$, so after n years, our equation would be:

$$B = 100(1 + 0.50)^n$$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential equations.

Summary of Exponential Growth or Decay

An **exponential growth or decay equation** is an equation that grows or shrinks at a constant percent growth/decay rate. An exponential equation has the form

$$y = a \cdot b^x \text{ where}$$

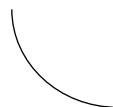
a is the initial or starting value of the equation (value of y when $x = 0$)

b is the growth factor (or decay factor)

If $b > 1$, then the graph is increasing (growth).



If $0 < b < 1$, then the graph is decreasing (decay).

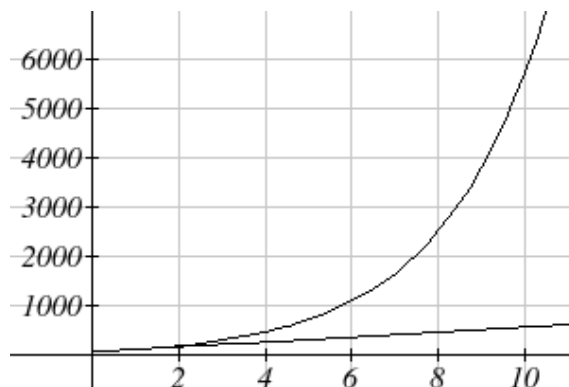


If r is the percent growth or decay rate (in decimal form), the exponential equation can be written as follows:

$$\text{Growth: } y = a(1 + r)^x \quad \text{or} \quad \text{Decay: } y = a(1 - r)^x$$

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company A and B described above over a longer time frame if the growth patterns were to continue.

Years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767



Example 15: Complete the table below.

Equation	Initial Value	Growth or Decay	Rate (as a %)
$y = 430(1.15)^x$			
$y = 23(0.8)^x$			
	1.87	Growth	5%
	52	Decay	10%

Solution: The completed table is below with the missing values in bold.

Equation	Initial Value	Growth or Decay	Rate (as a %)
$y = 430(1.15)^x$	430	Growth	15%
$y = 23(0.8)^x$	23	Decay	20%
$y = 1.87(1.05)^x$	1.87	Growth	5%
$y = 52(0.9)^x$	52	Decay	10%

Example 16: T represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured q quarters since January 1, 2010.

Interpret all of the parts of the equation $T = 86(1.64)^2 = 231.3056$.

Solution: Interpreting this from the basic exponential form,

- 1) We know that **86 is our initial value**. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts.
- 2) Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts **grows by 64%**.
- 3) $T = 231.3056$ means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,306 Android smart phone contracts.

Section 3.5A You Try Problems:

a) Given the three statements below, identify which represent exponential equations.

- 1) The cost of living allowance for state employees increases salaries by 3.1% each year
- 2) The value of an investment is increasing by \$300 each year
- 3) Tuition costs have increased by 2.8% each year for the last 3 years.

b) Looking at the following two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

$$A = 1,000(1.05)^t$$

$$B = 900(1.075)^t$$

c) Complete the table. The first one has been done for you.

Equation	Initial Value	Growth or Decay	Rate (as a %)
$y = 615(0.79)^x$	615	Decay	21%
$y = 500(1.03)^x$			
$y = 3200(1.2)^x$			
$y = 19,611(0.98)^x$			
	50	Growth	12%
	377	Decay	8%

Example 17: In the year 2008, India's population was 1.14 billion and growing at a rate of 1.34% per year. Write an exponential equation for India's population, and use it to predict the population in 2020.

Solution: Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134. Using the basic formula for exponential growth $y = a(1+r)^x$ we can write the formula,

$$P = 1.14(1 + 0.0134)^t \text{ or } P = 1.14(1.0134)^t$$

where P is the population of India (measured in billions) t years after 2008.

To estimate the population in 2020, we evaluate the equation at $t = 12$, since 2020 is 12 years after 2008.

$$P = 1.14(1.0134)^{12} \approx 1.337 \text{ billion people in 2020}$$

Answer: The model predicts there will be 1.337 billion people in 2020.

Example 18: Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution: With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is $a = 100$ mg, and our decay rate is 13%, so $r = 0.13$.

This gives the equation: $Q = 100(1 - 0.13)^d = 100(0.87)^d$

where Q is the amount (in mg) of Bismuth-210 remaining after d days. This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be

$$Q = 100(0.87)^7 = 37.73 \text{ mg}$$

Answer: After 7 days, 37.73 mg of Bismuth-210 remains.

Example 19: An investment is initially worth \$10,000. Write a formula for the value of this investment for each situation described below.

- The value increases by 5% every year.
- The value increases by 5% every 3 years.
- The value increases by 5% every 6 months.

Solution: In this problem, we are looking for three exponential equations of the form $V = ab^t$, where V represents the value of the investment (in dollars) after t years. Since the

initial value is \$10,000, we know that $a = 10,000$. In each case, the value of the investment is increasing by 5%, so $b = 1 + 0.05 = 1.05$. The only difference in these three cases is the *time period* for which the 5% increase occurs.

a) Since the 5% increase occurs every year, we have $V = 10,000(1.05)^t$

t	$V = 10,000(1.05)^t$
0	$V = 10,000(1.05)^0 = 10,000$
1	$V = 10,000(1.05)^1 = 10,500$
2	$V = 10,000(1.05)^2 = 11,025$
3	$V = 10,000(1.05)^3 = 11,576$

b) Since the 5% increase occurs every three years, we need to multiply by the growth factor, 1.05, only every three years. When $t = 3$ we multiply by the growth factor once. If $t = 6$, we multiply by the growth factor twice, and so on. The formula would then be $V = 10,000(1.05)^{t/3}$

t	$V = 10,000(1.05)^{t/3}$
0	$V = 10,000(1.05)^{0/3} = 10,000(1.05)^0 = 10,000$
3	$V = 10,000(1.05)^{3/3} = 10,000(1.05)^1 = 10,500$
6	$V = 10,000(1.05)^{6/3} = 10,000(1.05)^2 = 11,025$
9	$V = 10,000(1.05)^{9/3} = 10,000(1.05)^3 = 11,576$

c) Since the 5% increase occurs every six months, or twice each year, we need to multiply by the growth factor, 1.05, *two times* each year. The formula would then be $V = 10,000(1.05)^{2t}$

t	$V = 10,000(1.05)^{2t}$
0	$V = 10,000(1.05)^{2(0)} = 10,000(1.05)^0 = 10,000$
1/2	$V = 10,000(1.05)^{2(1/2)} = 10,000(1.05)^1 = 10,500$
1	$V = 10,000(1.05)^{2(1)} = 10,000(1.05)^2 = 11,025$
3/2	$V = 10,000(1.05)^{2(3/2)} = 10,000(1.05)^3 = 11,576$

To verify that these formulas are correct, we can compare the following tables of values:

t	a) Every Year	b) Every 3 Years	c) Twice a Year
0	10000	10000	10000
1	10500	10164	11025
2	11025	10331	12155
3	11576	10500	13401
4	12155	10672	14775
5	12763	10847	16289
6	13401	11025	17959

Example 20: Consider the following situations:

- 1) The population of a town is 83.6 thousand in the year 2005 and grows at a rate of 11.5% every 8 years.
- 2) A company's sales are \$14.2 million at the beginning of 2011, and have been decreasing at a rate of about 3.2% per quarter.

Write a formula for each of these situations using time, *in years*, as the input quantity

Solution:

- 1) In this situation, let P represent the population of the town (in thousands) t years after 2005. Since the population is increasing at a rate of 11.5% *every 8 years*, the formula would then be:

$$P = 83.6(1 + 0.115)^{t/8}$$

$$P = 83.6(1.115)^{t/8}$$

- 2) In this situation, let S represent the sales (in millions of dollars) t years after 2011. Since sales are decreasing at a rate of 3.2% *per quarter, or 4 times each year*, the formula for S would then be:

$$S = 14.2(1 - 0.032)^{4t}$$

$$S = 14.2(0.968)^{4t}$$

Section 3.5B You Try Problems:

An investment is initially worth \$84,300. Write a formula for the equation V , representing the value of this investment after t years in each of the following situations.

- a) The value increases by 6.5% every year.
- b) The value decreases by 3% every year.
- c) The value increases by \$600 every year.
- d) The value decreases by \$1400 every year.
- e) The value increases by 26% every 2 years.
- f) The value decreases by 13% every month.

Chapter 3 – Answers to You Try Problems3.1: Fraction Decimal Percent

a) $\frac{3}{5}$ 0.6 60%

b) $\frac{2}{100} = \frac{1}{50}$ 0.02 2%

c) $\frac{72}{100} = \frac{18}{25}$ 0.72 72%

d) $\frac{25}{1000} = \frac{1}{40}$ 0.025 2.5%

e) $4\frac{1}{2}$ 4.5 450%

3.2A: 50.2%

3.2B: 210

3.2C: 3,269mg, or 3.269 g

3.3: 2001-2002: Absolute Change=\$0.43 trillion Relative Change = 7.45% increase
 2005-2006: Absolute Change=\$0.54 trillion Relative Change = 6.83% increase
 2005-2006 saw a larger absolute increase, but a smaller relative increase.

3.4: a) \$97.98

b) \$62.90

3.5A: a) 1 & 3 are exponential equations; they grow by a % not a constant number.

b) B is growing faster, but after 3 years A still has a higher account balance

c)	Equation	Initial Value	Growth or Decay	Rate (as a %)
	$y = 615(0.79)^x$	500	Growth	3%
	$y = 500(1.03)^x$	3,200	Growth	20%
	$y = 3200(1.2)^x$	19,611	Decay	2%
	$y = 50(0.98)^x$	50	Growth	12%
	$y = 377(0.92)^x$	377	Decay	8%
	$y = 500(1.03)^x$	500	Growth	3%
	$y = 3200(1.2)^x$	3,200	Growth	20%

3.5B:

a) $V = 84,300(1.065)^t$

b) $V = 84,300(0.97)^t$

c) $V = 84,300 + 600t$

d) $V = 84,300 - 1,400t$

e) $V = 84,300(1.26)^{t/2}$

f) $V = 84,300(0.87)^{12t}$

CHAPTER 3 – PRACTICE PROBLEMS

- 1) Write each as a percent
 - a) $\frac{3}{4}$
 - b) $\frac{4}{5}$
 - c) $\frac{7}{10}$
- 2) The fraction $\frac{5}{7}$ is equivalent to what percent? Round your answer to the nearest tenth of a percent.
- 3) The fraction $\frac{7}{9}$ is equivalent to what percent? Round your answer to the nearest tenth of a percent.
- 4) The fraction $\frac{2}{11}$ is equivalent to what percent? Round your answer to the nearest tenth of a percent.
- 5) Write each as a percent:
 - a) 0.23
 - b) 0.215
 - c) 1.09
- 6) Write each as a decimal:
 - a) 15%
 - b) 4%
 - c) 7.5%
- 7) 20% is equivalent to what fraction in reduced terms?
- 8) 86% is equivalent to what fraction in reduced terms?
- 9) 58% is equivalent to what fraction in reduced terms?
- 10) What number is 70% of 40?
- 11) What number is 45% of 155?
- 12) 105 is what percent of 225?
- 13) 170 is what percent of 245?

- 14) 18 is 30% of what number?
- 15) 130 is 65% of what number?
- 16) Valerie paid \$170 for an item that was originally priced at \$550. What percent of the original price did Valerie pay? Round your answer to the nearest tenth of a percent.
- 17) Trader Joe's sold 8,132 bags of tortilla chips last month. If 7,687 of these bags were fat free, find the percent that were fat free. Round your answer to the nearest percent.
- 18) 76% of the questions on a student's test were correct. There were 50 questions. How many of the questions were correct?
- 19) At a restaurant, the bill comes to \$35. If you decide to leave a 15% tip, how much is the tip? Round your answer to the nearest cent.
- 20) What is the sales tax on a suit priced at \$295 if the sales tax is 7%?
- 21) What is the sales tax on a suit priced at \$1266 if the sales tax is 6%?
- 22) A lender requires a minimum down payment of 16% of the value of the home. You have \$39,000 cash available to use as a down payment toward a home. Determine the maximum home value that you can finance.
- 23) The population of a town increased from 3,850 in 2005 to 5,900 in 2012. By what percent did the population increase?
- 24) The price of a latte dropped from \$3.12 to \$3.00. Find the absolute change and relative change of the price of a latte.
- 25) Theresa's annual income increased from \$31,000 to \$48,000. Find the absolute change and relative change of Joe's annual income.
- 26) A shirt that was originally marked as \$24.95 rings up at the register as \$19.96. What is the percent discount on the shirt?
- 27) The U.S. Weather Bureau has a station on Mauna Loa in Hawaii that has measured carbon dioxide levels since 1959. At that time, there were 307 parts per million of carbon dioxide in the atmosphere. In 2005, the figure was 391 parts per million. Find the increase in carbon dioxide levels and the percent of increase. Round to the nearest tenth of a percent.
- 28) Anthony and his son Johnny are exercising more regularly and keeping track of their weight loss. Both Anthony and Johnny lost 10 pounds.
 - a) From this information alone, can you determine if they both lost the same percentage of weight?
 - b) Suppose Anthony's starting weight is 190 pounds and Johnny's starting weight is 155. Who lost a larger percent of their starting weight?

- 29) A company calls to notify you that the interest rate on your credit card will increase from 16% to 18%.
- a) Determine the number of percentage points the interest rate increased by.
 - b) Determine the percent increase of the interest rate.
- 30) The sales tax rate for the state of Washington was 5.7%. If the sticker price of a car in Washington is \$4,100, what will be the final cost of the car, including tax?
- 31) A store is having a 35% off sale. If the original price of the shirt is \$18.00, what is the price of the shirt after the discount is applied (not including tax)?
- 32) Hank currently makes \$13.00 per hour and is going to receive a 7% raise. What will be Hank's new hourly wage?
- 33) At a restaurant, the bill comes to \$50. If you decide to leave a 15% tip, what is the total amount you paid?
- 34) At a restaurant, the bill comes to \$32.20. If you decide to leave an 18% tip, what is the total amount you paid?
- 35) Sam lost 11% of his body weight after including healthier foods in his diet and exercising regularly. If his original weight was 218 lbs., what is his current weight?
- 36) A sales clerk received an increase in her annual salary, which is now \$27,300. If she received a 5% raise, what was her starting salary? Round your answer to the nearest cent.
- 37) Joyce paid \$84.50 for an item at the store that was 35% off the original price. What was the original price? Round your answer to the nearest cent.
- 38) You purchased a new car for a total cost of \$19,241.94, which includes the 13% sales tax. What was the original price of the car? Round your answer to the nearest cent.
- 39) Given the equation, $y = 24(0.72)^x$ find y for the following values of x .
- a) -3
 - b) 0
 - c) 2
- 40) Given the equation, $y = 5(1.08)^x$, find y for the following values of x .
- a) -1
 - b) 0
 - c) 3

College Math – Chapter 3 Practice Problems

41) For the equation $y = 170(0.8)^x$

- a) Identify the initial value
- b) Identify the behavior as growth or decay
- c) Identify the growth/decay factor
- d) Identify the growth/decay rate (as a percent)

42) For the equation $y = 5(2)^x$

- a) Identify the initial value
- b) Identify the behavior as growth or decay
- c) Identify the growth/decay factor
- d) Identify the growth/decay rate (as a percent)

43) For the equation $y = 110(1.3)^t$

- a) Identify the initial value
- b) Identify the behavior as growth or decay
- c) Identify the growth/decay factor
- d) Identify the growth/decay rate (as a percent)

44) For the equation $y = 60(0.7)^x$

- a) Identify the initial value
- b) Identify the behavior as growth or decay
- c) Identify the growth/decay factor
- d) Identify the growth/decay rate (as a percent)

45) Write the exponential equation with an initial value of 110 and a growth rate of 3.5%.

46) Write the exponential equation with an initial value of 1504 and a growth rate of 12%.

47) Write the exponential equation with an initial value of 63 and a decay rate of 28%.

48) Write the exponential equation with an initial value of 545 and a decay rate of 0.4%.

49) Write the exponential equation with an initial value of 79 and a growth rate of 119%.

50) Given the three statements below, identify which represent exponential change.

- a) The number of pennies in your piggy bank doubles every year.
- b) The value of a car depreciates at a rate of 22% per year.

- c) The temperature is decreasing at the rate of 3°F per hour.
- 51) When a new charter school opened in 2010, there were 970 students enrolled. Write an equation representing the number of students N attending this charter school t years after 2010, assuming that the student population:
- a) Increases by 73 students per year
 - b) Decreases by 36 students per year
 - c) Decreases by 9.4% per year
 - d) Increases by 6.1% per year
 - e) Remains constant (does not change)
- 52) A population numbers 1,944 organisms initially and increases by 5.2% each year.
- a) Write an exponential equation for the population of the organisms.
 - b) Predict the number of organisms after 7 years.
- 53) A population numbers 17,000 organisms initially and decreases by 8% each year.
- a) Write an exponential equation for the population of the organisms.
 - b) Predict the number of organisms after 2 years.
- 54) In 2014 Rose invested \$16,000 in a savings account for her newborn son. The account pays 3.6% interest each year. Determine the total value of the account in the year 2032, when her son will go to college. Round your answer the nearest cent.
- 55) You have saved \$1,000 to invest in a savings account. Write an exponential equation for the value of this investment for each situation that follows:
- a) The value increases by 4% every year.
 - b) The value increases by 4% every 2 years.
 - c) The value increases by 4% every 6 months.
 - d) The value increases by 4 % every 3 months.
- 56) An investment is initially worth \$7,400. Write an exponential equation representing the value of this investment after t years in each of the following situations.
- a) The value increases by 3.6% every year.
 - b) The value decreases by 2% every year.
 - c) The value increases by 11% every 3 years.
 - d) The value decreases by 5% every 6 months.

College Math – Chapter 3 Practice Problems

- 57) A new home is purchased for \$150,000. The value of the home increases by 11% every 2 years. Write an exponential equation for the value of the home using time *in years* as the input quantity.
- 58) A new home is purchased for \$200,000. The value of the home increases by 1.5% every 5 years. Write an exponential equation for the value of the home using time *in years* as the input quantity.
- 59) A new home is purchased for \$195,000. The value of the home decreases by 4% every 3 years. Write an exponential equation for the value of the home using time *in years* as the input quantity.
- 60) A new home is purchased for \$420,000. The value of the home decreases by 8% every 4 years. Write an exponential equation for the value of the home using time *in years* as the input quantity.

CHAPTER 4 – MODELING

Chapter 4 Learning Objectives:

- 1) Use technology to create a scatterplot and find a linear regression equation
- 2) Interpret the meaning of the slope and initial value (vertical intercept) of a regression equation
- 3) Use linear regression to predict the output for a given input
- 4) Use technology to calculate the correlation
- 5) Interpret the correlation from a value of r , or by looking at a scatterplot
- 6) Use exponential regression to predict the output for a given input

Sections

Section 4.1 – Linear Regression

Section 4.2 – Correlation

Section 4.3 – Exponential Regression

Section 4.4 – Modeling Applications

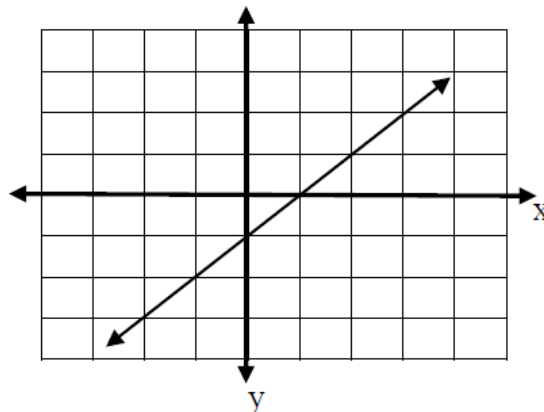
SECTION 4.1: LINEAR REGRESSION

In Chapter 2 we explored linear equations. Recall that linear equations have a constant rate of change, and can be written in the form $y = b + mx$, or $y = mx + b$. When graphed, a linear equation is a straight line, where all points that satisfy the linear equation fall exactly on the graph of the line. This can be seen in the graph below:

This is the graph of $y = -1 + x$.

Some of the points on this line are:
 $(-2, -3)$, $(0, -1)$, $(1, 0)$, and $(3, 2)$.

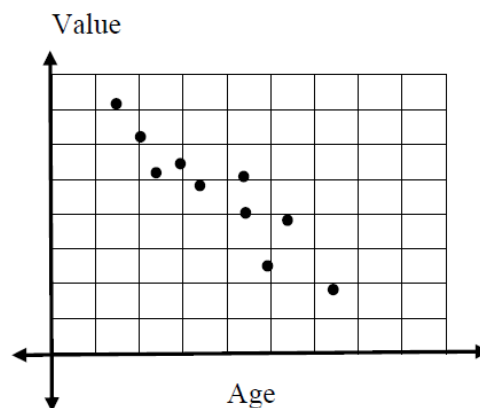
There are infinitely many other points on this line too.



Now consider the following graph that represents the relationship between the age of a vehicle (in years) and its value (in \$1000s) for 10 different vehicles.

Can we write a linear equation that represents this set of points? No, not exactly.

There is no straight line that will go through ALL of these points, but there does appear to be a linear relationship between the two variables.



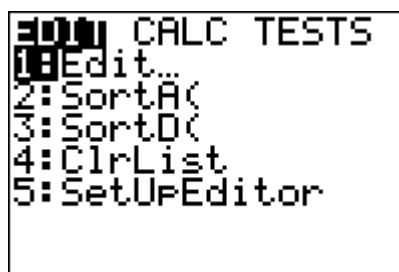
The graph that you see above is called a scatterplot. A scatterplot is used to show the relationship between two numerical variables. The input (independent, or predictor) variable is plotted on the horizontal axis, and the output (dependent, or response) variable is plotted on the vertical axis.

When data points have a linear trend, but do not all land perfectly on a straight line, we can use our calculators to find the equation of the line that “best fits” the data set. This line is called the linear regression equation. **Note:** the regression equation will not go through every data point, but it will approximate the trend of the data points.

Time, t , in weeks	0	1	2	3	4	5
Weight, W , in pounds	196	192	193	190	190	186

Draw a scatterplot of the data. Does the data appear linear?

Step 1: Enter the data into your calculator



- Press STAT (Second Row of Keys)
- Press ENTER to access 1:Edit under EDIT menu
- **Note:** *Be sure all data columns are cleared. To do so, use your arrows to scroll up to L1 or L2 then click CLEAR then scroll down. (DO NOT CLICK DELETE!)*

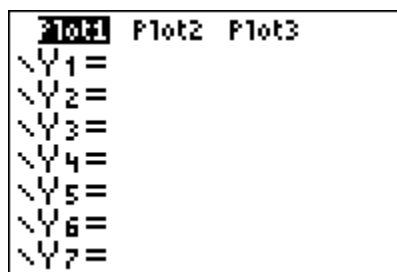
Once your data columns are clear, enter the input data into L1 (press ENTER after each data value to get to the next row) then right arrow to L2 and enter the output data into L2. Your result should look like this when you are finished (for L1 and L2):

L1	L2	L3	1
	-----	-----	

L1(1) =

L1	L2	L3	2
0	196	-----	
1	192		
2	193		
3	190		
4	190		
5	186		
-----	-----		
L2(6) = 186			

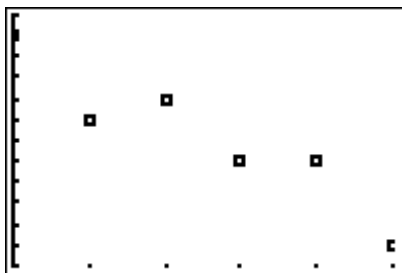
Step 2: Turn on your Stat Plot



- Press Y=
- Use your arrow keys to scroll up to Plot1
- Press ENTER
- Scroll down and Plot1 should be highlighted as at left
- Clear out all entries below

Step 3: Graph the Data in an Appropriate Viewing Window

Click the ZOOM key, and scroll down to Option 9: “ZoomStat”



- Once your cursor is on “ZoomStat”, press the GRAPH key. A graph of your data should appear in an appropriate window so that all data points are clearly visible.

The scatterplot above shows that the data points seem to have a linear relationship.

From the previous scatterplot, it looks like there is a negative linear relationship between the time in weeks and the weight in pounds. We can see this linear relationship because the points are scattered around an imaginary straight line. More specifically, as the time increases, the weight seems to decrease by a fairly consistent amount.

Section 4.1A You Try: Consider the data set below.

x	3	5	8	9	11	12	15
y	1.1	3.2	7	6.8	9.5	9.8	13

- Draw a scatterplot for the data set.
- Does the data set appear to be increasing or decreasing?
- Does the data set appear to have a linear relationship?

Just because data are not EXACTLY linear does not mean we cannot write an approximate linear model for the given data set. In fact, most data in the real world are NOT exactly linear and all we can do is write models that are close to the given values.

Once we have determined using a scatterplot that our two variables appear to have a roughly linear relationship, we can use a process called **linear regression** to find the equation of the line that best fits the data points. If you take a statistics class, you will learn a lot more about this process. In this class, you will be introduced to the basics. This process is also called “FINDING THE LINE OF BEST FIT”. This process is lengthy and calculation intensive by hand, but can be done quickly using our graphing calculator.

Our calculator will give us the best linear equation possible by taking into account ALL the given data points.

Note: Unless your data are exactly linear, the regression equation will not match all data points exactly. It is a model used to predict outcomes not provided in the data set

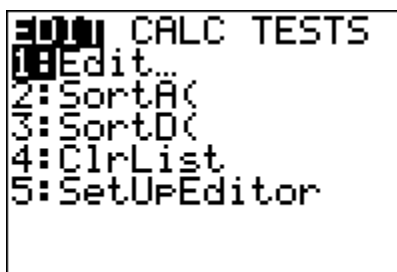
Example 2: Consider the data set from the previous problem:

Time, t , in weeks	0	1	2	3	4	5
Weight, W , in pounds	196	192	193	190	190	186

Use your graphing calculator to find the regression equation.

Step 1: Enter the data into your calculator

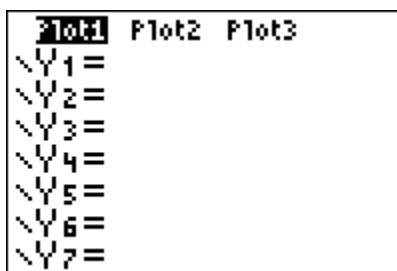
Press STAT then select option 1: Edit under EDIT menu. Clear lists, then enter the values.



L1	L2	L3	Z
0	196	-----	
1	192		
2	193		
3	190		
4	190		
5	186		
-----	-----		
L2(6) = 186			

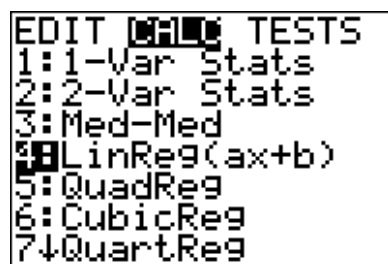
**** Note:** If you ever accidentally DELETE a column, then go to STAT, Option 5: SetUpEditor>ENTER. When you go back to STAT, your column should be restored.

Step 2: Turn on your Stat Plot and Graph the Data in an Appropriate Viewing Window (Refer to previous example for help)

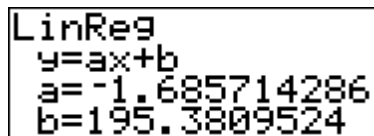


Note: Since we already graphed the scatterplot for this data set (and know the data looks linear), you could skip this step. Graphing the scatterplot is not required in order to calculate the regression equation, but you should always verify using a scatterplot that your data looks linear BEFORE doing a linear regression!

Step 3: Access the Linear Regression section of your calculator



- Press STAT
- Scroll to the right one place to CALC
- Scroll down to 4:LinReg(ax+b)
- Your screen should look as the one at left

Step 4: Determine the linear regression equation


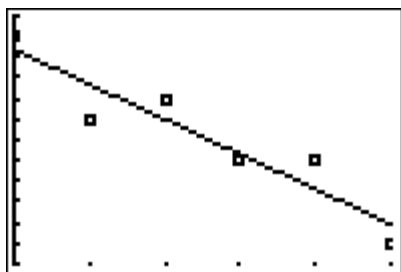
LinReg
y=ax+b
a=-1.685714286
b=195.3809524

- Press ENTER twice in a row to view the screen at left
- The calculator computes values for slope (a) and y-intercept (b) in what is called the equation of best-fit for your data.
- Identify these values and round to the appropriate places. Let's say 2 decimals in this case. So, $a = -1.69$ and $b = 195.38$
- Now, replace the a and b in $y = ax + b$ with the rounded values to write the actual equation: $y = -1.69x + 195.38$

To write the equation in terms of initial variables, we would write $W = -1.69t + 195.38$

Step 5: Graphing the Regression Equation on top of the Stat Plot

Once we have the equation figured out, it's nice to graph it on top of our data to see how things match up.



- Enter the Regression Equation with rounded values into Y=
- Press GRAPH
- You can see from the graph that the “best fit” line does not hit very many of the given data points. But, it will be the most accurate linear model for the overall data set.

Important Note: When you are finished graphing your data, TURN OFF YOUR PLOT1. Otherwise, you will encounter an INVALID DIMENSION error when trying to graph other equations. To do this:

- Press Y=
- Use your arrow keys to scroll up to Plot1
- Press ENTER
- Scroll down and Plot1 should be Un-highlighted

Section 4.1B You Try Problem:

Consider the data set below.

x	0	2	4	6	8	10	12
y	23.76	24.78	25.93	26.24	26.93	27.04	27.93

Use your graphing calculator to find the linear regression equation for this data set.

Once you have a linear regression equation, you can then use that equation to provide you with information about *how your two variables change together*, and to make *predictions* about other values of your variables of interest.

Example 3: Consider again the data set from the previous problem:

Time, t , in weeks	0	1	2	3	4	5
Weight, W , in pounds	196	192	193	190	190	186

- Interpret the meaning of the vertical intercept.
- Interpret the meaning of the slope.
- Use your regression equation to predict the weight in pounds after 2.5 weeks.
- Use your regression equation to predict the weight in pounds after 6 weeks.

Solution: Recall, the regression equation for this data set was:

```
LinReg
y=ax+b
a=-1.685714286
b=195.3809524
```

$$y = -1.69x + 195.38$$

To write the equation in terms of initial variables, we would write $W = -1.69t + 195.38$

a) The vertical intercept is the value of W when $t = 0$. Based on the regression equation, we know that when $t = 0$, $W = 195.38$. This tells us that when first starting the weight loss program, the individual weighed approximately 195.38 pounds.

Note: In this particular case, we had a data point for time 0 weeks of 196 pounds. In general, the regression equation gives an estimate (or predicted value) of the response variable. Output from the regression equation will rarely equal the exact data values themselves.

b) The slope of the regression equation is -1.69 . Recall that the slope measures $\frac{\text{change in } y}{\text{change in } x}$ or in our problem, $\frac{\text{change in weight } (W)}{\text{change in time } (t)}$. We can write the slope for this equation of -1.69 as a fraction by writing it over 1:

$$\frac{\text{change in weight } (W)}{\text{change in time } (t)} = \frac{-1.69 \text{ pounds}}{1 \text{ week}}$$

To interpret the meaning of the slope, we would say that for each additional week, the weight decreases by approximately 1.69 pounds.

c) To predict the weight in pounds after 2.5 weeks, plug in $t = 2.5$ to the regression equation, and solve for W .

$$W = -1.69(2.5) + 195.38$$

$$W = 191 \text{ pounds}$$

d) To predict the weight in pounds after 6 weeks, plug in $t = 6$ to the regression equation, and solve for W .

$$W = -1.69(6) + 195.38$$

$$W = 185 \text{ pounds}$$

In the previous problem, we used our regression equation to tell us how our two variables change together, and to make predictions about other values. You may have noticed that we made predictions about the values of our dependent variable only for values of our independent variable that were close to the actual values in our data set. In other words, our original data set included values of time from 0 – 5 weeks. Based on that information, we created our model. It would not be appropriate to try to make predictions for values of time that are far outside our original range of time values (from 0 – 5 weeks). This is called **extrapolation** and can result in very poor predictions. Consider the following example.

Example 4: Consider again the data set from the previous problem:

Time, t, in weeks	0	1	2	3	4	5
Weight, W, in pounds	196	192	193	190	190	186

Would it be reasonable to use your regression equation to predict the weight in pounds after 100 weeks? Why or why not?

Solution: Recall, the regression equation for this data set was

$$W = -1.69t + 195.38$$

Our model was created based off of data from 0 to 5 weeks into a weight loss program. Is it reasonable to assume that the same pattern of weight loss that held in weeks 0-5 would be present in week 100? It certainly doesn't seem so. This is an example of extrapolation. Let's see what would happen if we tried to use the model to predict the weight in pounds after 100 weeks. We plug in $t = 100$ to the regression equation, and solve for W .

$$W = -1.69(100) + 195.38$$

$$W = 26 \text{ pounds}$$

If this pattern of weight loss continued, the individual would weigh only 26 pounds after 100 weeks! This is not likely to be an accurate prediction.

Section 4.1C You Try Problem:

The following table gives the total number of live Christmas trees sold, in millions, in the United States from 2004 to 2011. (Source: Statista.com).

Year	2004	2006	2008	2010	2011
Total Number of Christmas Trees Sold in the U.S. (in millions)	27.10	28.60	28.20	27	30.80

Use your calculator to determine the equation of the *regression line*, $C(t)$ where t represents the number of years since 2004.

Start by entering new t values for the table below based upon the number of years since 2004. The first few are done for you:

t = number of years since 2004	0	2			
C = Total number of Christmas trees sold in the U.S. (in millions)	27.10	28.60	28.20	27	30.80

- Identify the slope of the regression equation and explain its meaning in the context of this problem.
- Use the regression equation to predict the number of Christmas trees that will be sold in the year 2013. Write your answer as a complete sentence.
- Should you use your regression equation to predict the number of Christmas trees that will be sold in the year 2030? Why or why not?

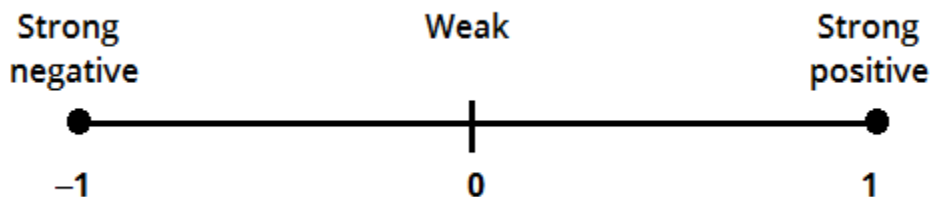
SECTION 4.2: CORRELATION

In the last section we learned about relationships between two variables that appeared to be linear. In some cases, the scatterplot shows very clearly that the two variables are linearly related, but in many cases it can be a bit challenging to tell exactly how strong the linear relationship is between the two variables of interest. The **linear correlation (r)**, also called the **correlation coefficient**, is a way to numerically measure the *strength* and the *direction* of the linear relationship between two variables.

Properties of Correlation:

- 1) We use “ r ” to represent the correlation
- 2) $-1 \leq r \leq 1$
- 3) The sign of r indicates the direction of the linear relationship between the 2 variables
 - If $r < 0$, there is a negative linear relationship between the 2 variables (slope of the regression equation is negative).
 - If $r > 0$, there is a positive linear relationship between the 2 variables (slope of the regression equation is positive).
- 4) The magnitude of r indicates the strength of the linear relationship between the 2 variables
 - If $r \approx 0$, then there is no linear relationship between the 2 variables
 - If $r \approx |1|$ then there is a very strong linear relationship between the 2 variables
 - If $r = 1$ or -1 , then there is an exact linear relationship between the 2 variables (every data point falls exactly on the regression line)

These properties of correlation can be summarized in the graphic below:



Example 5: Consider again the data set from the previous problem:

Consider the following scatterplots, and their corresponding correlation values.

a) $r = 0.94$



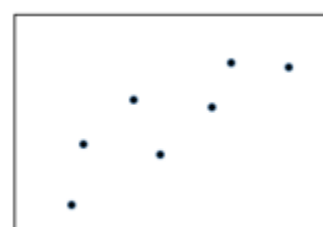
b) $r = -0.15$



c) $r = -0.88$



d) $r = 0.58$



Classify the strength and direction of the linear relationship between the 2 variables in each scatterplot.

Solution:

- a) Here $r = 0.94$, which is close to positive 1. We would say these 2 variables have a strong, positive linear relationship.
- b) Here $r = -0.15$, which is negative and close to 0. We could say these 2 variables have a very weak, negative linear relationship.
- c) Here $r = -0.88$, which is close to negative 1. We would say these 2 variables have a strong, negative linear relationship.
- d) Here $r = 0.58$, which is positive, but not particularly close to 0 or 1. We would say these 2 variables have a moderate, positive linear relationship.

Note: There is no “cutoff” value for which values of r are considered “strong” and which are considered “weak”. The interpretation of r can be fairly subjective. What you should know is that the farther the value of r is from 0, the stronger the linear relationship between the 2 variables is considered to be.

Now that we have a basic understanding of interpreting the correlation, we can discuss how to find it. As with linear regression, the calculations for finding correlation by hand are rather time consuming. We will again rely on the graphing calculator to find the correlation for us.

Example 6: Consider again the data set from the weight loss problem:

Time, t, in weeks	0	1	2	3	4	5
Weight, W, in pounds	196	192	193	190	190	186

- a) Use your graphing calculator to find the correlation between time and weight.
- b) Interpret the meaning of the correlation you found in part a.

Solution: To find the correlation on the graphing calculator, press the “2nd” key and then “Catalog”. “Catalog” is a 2nd function on the zero key. Use the arrow keys to scroll down to “DiagnosticOn”. Hit ENTER. On the main screen, your calculator should display DiagnosticOn followed by a blinking cursor. Hit ENTER. The calculator should display “Done”. Once you have turned on the DiagnosticON, you do not have to do it again. Also, it will not interfere with any other calculator functions, so you can just leave it on.

Once you have turned on “DiagnosticOn” you can follow the instructions for finding the linear regression equation from section 3.3. When the calculator displays the linear regression equation, it will also display the correlation value, r .

- a) Enter your data into L1 and L2, then press STAT, scroll over to CALC, and select option 4: LinReg(ax+b). Your output should consist of the values of a and b (just like we did in section 3.3) AND two new values: r^2 and r. The value of $r = -0.94$. (**Note:** The quantity r^2 is the coefficient of determination, which we do not cover in this course).
- b) The value of $r = -0.94$. This means that time and weight have a strong, negative linear relationship. This means that as time increases, weight tends to decrease. This can be confirmed by a scatterplot of the data which showed a negative slope to the data points.

Section 4.2 You Try:

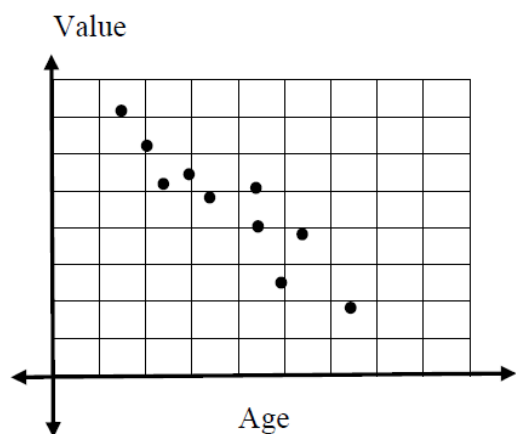
Recall, the following table gives the total number of live Christmas trees sold, in millions, in the United States from 2004 to 2011. (Source: Statista.com).

t = number of years since 2004	0	2	4	6	7
C = Total Number of Christmas Trees Sold in the U.S. (in millions)	27.10	28.60	28.20	27	30.80

- a) Use your graphing calculator to find the numerical value of the correlation.
- b) Interpret the meaning of the correlation from part a).

SECTION 4.3: EXPONENTIAL REGRESSION

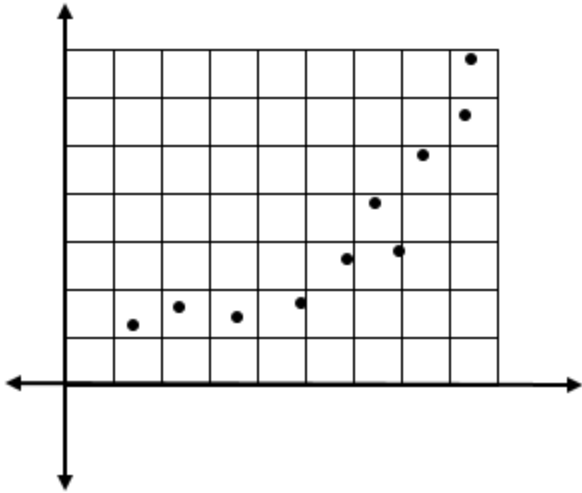
In a previous section we looked at the following graph that represents the relationship between the age of a vehicle (in years) and its value (in \$1000s) for 10 different vehicles.



We used linear regression to find the equation of the line that best models the relationship between Age and Value of vehicles from this data set.

We did **linear** regression because a scatterplot of the points showed the points were scattered about a straight **line**.

In this section, we will learn how to apply the technique of regression to data that is not linear, but instead exponential. Consider the following scatterplot:



Notice that the points on this graph **do not** look linear (they don't fall along a straight line) but they definitely fall along a curve.

The shape of this curve is exponential, so we will use our calculator to find the equation of the exponential curve that best fits this data set.

This process is called exponential regression.

As a reminder, the graph that you see above is called a **scatterplot**. The **input (independent, or predictor) variable** is plotted on the horizontal axis, and the **output (dependent, or response) variable** is plotted on the vertical axis.

Just as with linear equations, we can work with a data table and, if appropriate, model that data using **exponential regression**. The steps are almost the same as those followed for linear regression.

Example 7: The table below shows the population, P , in a given state after t years.

t (years)	P (Population)
5	5,234,456
10	4,892,345
15	4,012,345

Use the Exponential Regression feature of your calculator to generate a mathematical model for this situation. Round "a" to the nearest whole number and "b" to 3 decimals.

Step 1: Press STAT>EDIT>ENTER to show data entry area. The STAT button is on the second row, third column.

L1	L2	L3
██████	-----	-----
L1() =		

- Data entry area should be blank to begin.
- To clear, go column by column.
- Scroll to column header using the arrow keys then press Clear>Enter.
- Use the arrow keys to move back and forth.

Step 2: Enter the values in the first column of your data set, t , into L1. Enter the values in the second column of your data set, P , into L2.

L1	L2	L3	Z
5	5.23E6	-----	
10	4.88E6		
15	4.00E6		
-----	-----		
L2(3) = 4012345			

Note: The numbers for L2 are displayed using scientific notation (the E notation) since they are too long for the column width. Scroll to each number and see its full representation in the bottom of the screen. See example highlighted at right.

Step 3: Press STAT>CALC>0:ExpReg>ENTER>ENTER

```

EDIT  [2nd] [F5] TESTS
5↑QuadReg
6: CubicReg
7: QuartReg
8: LinReg(a+bx)
9: LnReg
0: ExpReg
↓PwrReg
  
```

Output

```

ExpReg
y=a*b^x
a=6110389.818
b=.9737616611
  
```

Step 4: Thus, your exponential equation (with values rounded as the problem indicates) is

$$P = 6110390(0.974)^t$$

Example 8: Determine the exponential regression equation that models the data below:

t (in years)	0	1	2	4	6	9	12	16
P (population)	125	75	50	32	22	16	10	5.7

When you write your final equation, round “ a ” to 1 decimal place and “ b ” to three decimal places.

a) Write the exponential regression equation in the form $y = ab^x$

b) What is the rate of decay (as a %) for this equation? Interpret the meaning of your answer.

c) Determine P when $t = 20$. Show your work, and interpret the meaning of your answer. Round to two decimal places.

Solution:

a) **Step 1:** Press STAT>EDIT>ENTER to enter the values for t into L1 and the values for P into L2.

L1	L2	L3
2	50	
4	32	
6	22	
9	16	
12	10	
16	5.7	

L2(8) = 5.7		

Step 2: Once those values are entered, find the regression equation by pressing
STAT>CALC>0:ExpReg>ENTER>ENTER

```
ExpReg
y=a*b^x
a=82.94282346
b=.836691901
```

Answer: Thus, your exponential regression equation is:

$$P = 82.9(0.837)^t$$

b) Recall, $b = 1 - r$ for exponential decay, so to find the rate of decay r we need the difference between 1 and 0.837.

$$r = 1 - 0.837 = 0.163$$

Then convert the rate r into a percentage by multiplying by 100.

Answer: The rate of decay is 16.3%. The population is decreasing by 16.3% each year.

c) To determine P when $t = 20$, use the regression equation from part a. Substitute $t = 20$, and solve for P .

$$P = 82.9(0.837)^{20} \approx 2.36$$

Answer: After 20 years, the population will be approximately 2.36.

Section 4.3 You Try:

The table below shows the value, V , of an investment (in thousands of dollars) after n years.

n	0	3	5	10	15	22
V	4.63	5.92	6.88	10.23	15.21	26.39

- Use your calculator to determine the exponential regression equation that models the set of data above. Round the " a " value to two decimals, and round the " b " value to three decimals. Use the indicated variables.
- Based on the equation found in part a), at what percent rate is the value of this investment increasing each year?
- Determine V when $n = 12$, and write your answer in a complete sentence. Round your answer to two decimal places.

SECTION 4.4: MODELING APPLICATIONS

Now that you have seen examples of both linear and exponential regression, we will look at real world applications involving mathematical modeling. In these cases, you must first make a scatterplot of your data to determine whether linear regression, exponential regression, or neither appear to be appropriate. If the data appear to be either linear or exponential based on your scatterplot, then you will go on to create a mathematical model using the data.

Example 9: Scientists are studying the growth of a particular virus in a large community. The table below shows the case count, C , in the given community after d days.

d (Days)	0	2	3	5	7	8	10	12	13
C (Case Count)	201	211	220	231	272	293	329	417	484

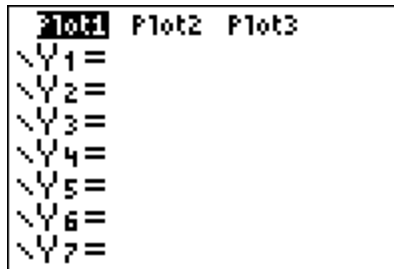
- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine C when $d = 15$, and write your answer in a complete sentence. Round your answer to three decimal places.

Solution:

a) Press STAT>EDIT>ENTER to enter the values for t into L1 and the values for P into L2.

L1	L2	L3
0	201	---
2	211	
3	220	
5	231	
7	272	
8	293	
10	329	

Once those values are entered, turn on your stat plot:



- Press Y=
- Use your arrow keys to scroll up to Plot1
- Press ENTER
- Scroll down and Plot1 should be highlighted as at left
- Clear out all entries below

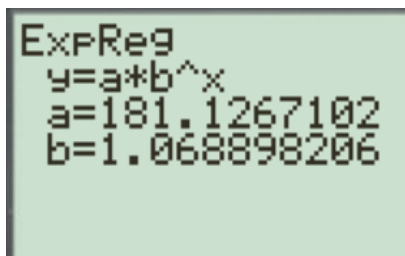
Click the ZOOM key, and scroll down to Option 9: "ZoomStat"



- Once your cursor is on "ZoomStat", press the GRAPH key. A graph of your data should appear in an appropriate window so that all data points are clearly visible.

The scatterplot above shows that the data points seem to have an **exponential** relationship.

b) Since the scatterplot showed an exponential relationship, you can find the exponential regression equation by pressing: STAT>CALC>0:ExpReg>ENTER>ENTER



The exponential regression model is:

$$C = 181.127(1.069)^d$$

c) To find C when $d = 15$, plug in $d = 15$ in your exponential regression model:

$$C = 181.127(1.069)^{15}$$

$$C = 492.775$$

After 15 days, the case count of the virus in this community is approximately 493 cases.

Example 10: A local real estate company has decided to track the number of calls they receive each month requesting information on a home. The table below shows the number of calls, N , received each month, m , for a period of 12 months.

m (Month)	1	2	3	4	5	6	7	8	9	10	11	12
N (Number of Calls)	1204	1990	2707	3221	5409	4983	4632	5714	2293	1998	1719	532

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables. Otherwise, state “not appropriate”.

Solution:

- Press STAT>EDIT>ENTER to enter the values for m into L1 and the values for N into L2.

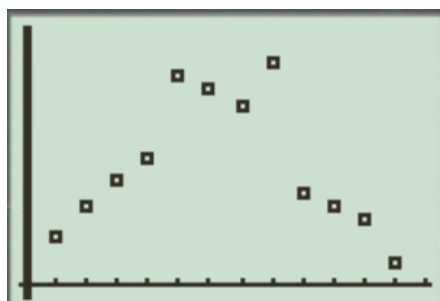
L1	L2	L3
1	1204	---
2	1990	
3	2707	
4	3221	
5	5409	
6	4983	
7	4632	

Once those values are entered, turn on your stat plot:

Plot1	Plot2	Plot3
Y1=		
Y2=		
Y3=		
Y4=		
Y5=		
Y6=		
Y7=		

- Press Y=
- Use your arrow keys to scroll up to Plot1
- Press ENTER
- Scroll down and Plot1 should be highlighted as at left
- Clear out all entries below

Click the ZOOM key, and scroll down to Option 9: “ZoomStat”



- Once your cursor is on “ZoomStat”, press the GRAPH key. A graph of your data should appear in an appropriate window so that all data points are clearly visible.

The scatterplot above shows that the data points do not have a either a linear or an exponential relationship. So it is not appropriate to make either of the regression models.

Example 11: The table below shows the square footage, s , and sale price (in thousands of dollars), P , for several homes in Scottsdale, Arizona.

s (square feet)	1517	1790	2315	2940	3300
P (Price)	399	433	476	564	576

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine P when $s = 3450$, and write your answer in a complete sentence. Round your answer to three decimal places.

Solution:

- Press STAT>EDIT>ENTER to enter the values for s into L1 and the values for P into L2

L1	L2	L3
1517	399	--
1790	433	--
2315	476	--
2940	564	--
3300	576	--
-----	-----	-----

Once those values are entered, turn on your stat plot:

2nd Y=	Plot2	Plot3
$\backslash$ Y1=		
$\backslash$ Y2=		
$\backslash$ Y3=		
$\backslash$ Y4=		
$\backslash$ Y5=		
$\backslash$ Y6=		
$\backslash$ Y7=		

- Press Y=
- Use your arrow keys to scroll up to Plot1
- Press ENTER
- Scroll down and Plot1 should be highlighted as at left
- Clear out all entries below

Click the ZOOM key, and scroll down to Option 9: "ZoomStat"



- Once your cursor is on "ZoomStat", press the GRAPH key. A graph of your data should appear in an appropriate window so that all data points are clearly visible.

The scatterplot above shows that the data points appear to have a **linear** relationship.

b) Next fit a linear model to this data set.

```
LinReg
y=ax+b
a=.1035170426
b=244.0161682
```

The linear regression model is: $P = 0.014s + 244.016$

c) To find P when $s = 3450$, plug in $s = 3450$ in your linear regression model:

$$P = 0.014(3450) + 244.016$$

$$P = 602.816$$

A home in Scottsdale, Arizona that is 3450 square feet would have a sale price of approximately \$602,816.

Section 4.4 You Try Problem:

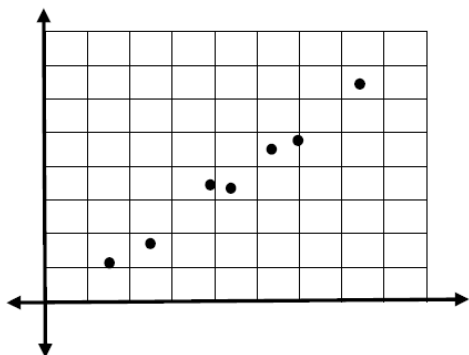
The table below shows the population, P , of geese who migrated through a certain location after d days.

d (in days)	0	2	6	12	18	21
P (population)	1508	976	404	138	31	14

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine P when $d = 10$, and write your answer in a complete sentence. Round your answer to three decimal places.

Chapter 4 – Answers to You Try Problems

4.1: a)



b) Increasing

c) Yes, there appears to be a linear trend

4.1B: $y = 0.32x + 24.16$

4.1C: a) $C = 0.279t + 27.28$

b) For each additional year since 2004, the number of Christmas trees sold in the U.S. will increase by approximately 279,000 trees.

c) 29.79 million Christmas trees

d) No. The original data is for years 2004 through 2011. The year 2030 is too far from the original data set.

4.2: a) $r = 0.52$

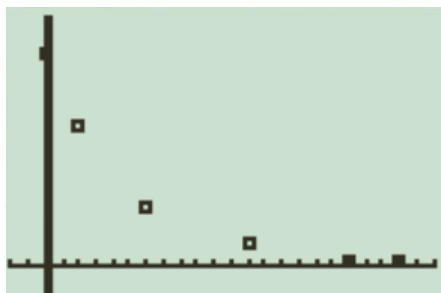
b) There is a moderate, positive linear relationship between the time in years since 2004 and the number of Christmas trees sold in the U.S.

4.3: a) $V = 4.64(1.082)^n$

b) The investment is increasing by 8.2% each year.

c) $V = 11.95$. After 12 years, the investment is worth approximately \$11,950.

4.4 a) The data appears to be exponential



b) $P = 1559.607(0.803)^d$

c) $P = 173.848$.

After 10 days, the population of geese migrating through this area is approximately 174 geese

CHAPTER 4 – PRACTICE PROBLEMS

1) Use the table below for question 1.

x	1	3	4	6	7	9	10
y	437	901	1155	1358	1768	2103	2437

- Use your graphing calculator to create of scatterplot of the data set shown below.
- Does the data set appear to be increasing or decreasing?
- Does the data set appear to be roughly linear?

2) Use the table below for question 2.

x	2	9	14	23	33	42	2
y	-60.2	-130.1	-243.7	-328.9	-580.5	-643.8	-60.2

- Use your graphing calculator to create of scatterplot of the data set shown below.
- Does the data set appear to be increasing or decreasing?
- Does the data set appear to be roughly linear?

3) Use the table below for question 3.

x	8	12	14	23	24	29	33
y	10	20	15	24	35	30	28

- Use your graphing calculator to create of scatterplot of the data set shown below.
- Does the data set appear to be increasing or decreasing?
- Does the data set appear to be roughly linear?

4) Use the table below for question 4.

x	20	24	27	32	48	60	68
y	12	20	28	41	24	15	11

- Use your graphing calculator to create of scatterplot of the data set shown below.
- Does the data set appear to be increasing or decreasing?
- Does the data set appear to be roughly linear?

5) Calculate the linear regression equation for the data below.

x	1	3	4	6	7	9	10
y	437	901	1155	1358	1768	2103	2437

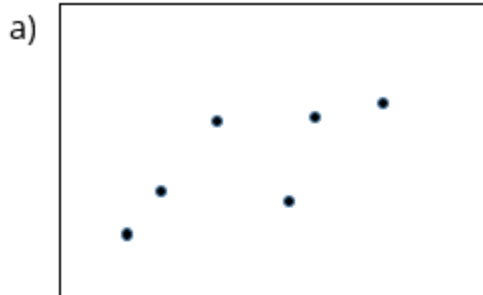
6) Calculate the linear regression equation for the data below

x	2	9	14	23	33	42	2
y	-60.2	-130.1	-243.7	-328.9	-580.5	-643.8	-60.2

7) Calculate the linear regression equation for the data below.

x	8	12	14	23	24	29	33
y	10	20	15	24	35	30	28

8) Consider the following 4 scatterplots. Match each scatterplot with the most appropriate value from the 4 correlation coefficients: $r = 0.38$, $r = 0.94$, $r = -0.15$, $r = -0.85$



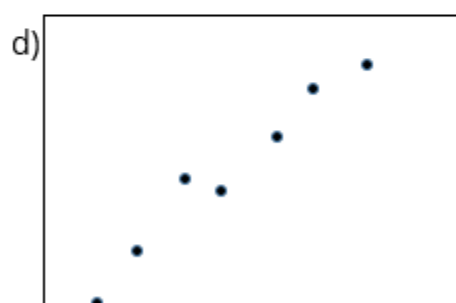
$r = \underline{\hspace{2cm}}$



$r = \underline{\hspace{2cm}}$



$r = \underline{\hspace{2cm}}$



$r = \underline{\hspace{2cm}}$

9) Find and interpret the linear correlation for the data in the table below.

x	1	3	4	6	7	9	10
y	437	901	1155	1358	1768	2103	2437

10) Find and interpret the linear correlation for the data in the table below.

x	2	9	14	23	33	42	2
y	-60.2	-130.1	-243.7	-328.9	-580.5	-643.8	-60.2

11) Find and interpret the linear correlation for the data in the table below.

x	8	12	14	23	24	29	33
y	10	20	15	24	35	30	28

12) Find and interpret the linear correlation for the data in the table below.

x	20	24	27	32	48	60	68
y	12	20	28	41	24	15	11

13) The following table shows the number of newspaper subscriptions in Middletown, USA where t represents the number of years since 2002 ($t = 0$ in 2002) and S represents the total subscriptions each year measured in thousands. Find and interpret the linear correlation.

t (years since 2002)	0	2	4	6	8
S (total subscriptions in 1000's)	448	372	198	145	45

14) Scott is hiking the Appalachian Trail from Georgia to Maine. The distance of his hike is 2200 miles. It took Scott 123 days to complete the hike. The data below represent the distance, D , he had hiked t days after the start of his trip. Find and interpret the linear correlation.

t (days hiking)	0	32	47	73	99	123
D (distance in miles)	0	590	912	1212	1876	2200

15) Which implies a stronger linear relationship, a correlation of -0.84 or -0.47 ? Explain.

16) Which implies a stronger linear relationship, a correlation of -0.73 or 0.51 ? Explain.

- 17) The following table shows the number of newspaper subscriptions in Middletown, USA where t represents the number of years since 2002 ($t = 0$ in 2002) and S represents the total subscriptions each year measured in thousands.

t (years since 2002)	0	2	4	6	8
S (total subscriptions in 1000's)	448	372	198	145	45

- Use your graphing calculator to create of scatterplot of the data set.
- Based on your graph above, do the data appear to be approximately linear?
- Use your calculator to determine the *regression equation* in $S = at + b$ form. (Round to 2 decimal places)
- What is the slope of your regression model for S and what is its meaning in the context of this problem?
- What is the initial value of your linear regression model for S and what is its meaning in the context of the problem.
- Use your linear regression equation to estimate the total number of subscriptions in 2007 (i.e. when $t = 5$). Show your computations here and your final result.
- Use your linear regression equation to estimate the total number of subscriptions in 2004. How does this value compare to the data value in the table?
- Should you use your linear regression equation to estimate the circulation in the year 2030? Why or why not?

- 18) Scott is hiking the Appalachian Trail from Georgia to Maine. The distance of his hike is 2200 miles. It took Scott 123 days to complete the hike. The data below represent the distance, D , he had hiked t days after the start of his trip.

t (days hiking)	0	32	47	73	99	123
D (distance in miles)	0	590	912	1212	1876	2200

- Use your graphing calculator to create of scatterplot of the data set.
- Based on your graph above, do the data appear to be approximately linear?
- Use your calculator to determine the *regression equation* in $D = at + b$ form. (Round to 2 decimal places)
- What is the slope of your regression model for D and what is its meaning in the context of this problem?
- Use your linear regression equation to estimate the total number of miles Scott has hiked in 50 days. Show your computations here and your final result.
- Should you use your linear regression equation to estimate when Scott has hiked 2500 miles? Why or why not?

19) Determine the exponential regression equation that models the data below. Write your final equation with " a " rounded to one decimal place and " b " to three decimal places

t	0	1	2	4	6	9	12	16
P	97	87	78	62	51	36	25	17

- Write exponential regression equation in the form $y = ab^x$. Use the indicated variables.
- What is the rate of decay (as a %) for this equation?
- Using your regression model, determine P when $t = 8$. Give your answer to 2 decimal places.

20) The table below shows the value, V , of an investment (in thousands of dollars) after n years.

n	0	3	5	10	15	22
V	3.74	4.58	5.24	7.46	10.21	17.01

- Use your calculator to determine the exponential regression equation that models the set of data above. Round the " a " value to two decimals, and round the " b " value to three decimals. Use the indicated variables.
- Based on your regression model, what is the percent increase per year?
- Find V when $n = 12$, and interpret its meaning in a complete sentence. Round your answer to two decimal places.

21) The following data represents the number of radioactive nuclei in a sample after t days. Round any answers to 3 decimal places.

$t = \text{time in days}$	0	1	4	6	9	12	17
$N = \text{number of nuclei}$	2500	2287	1752	1451	1107	854	560

- Use the exponential regression feature of your calculator to find the regression equation of the form $N = a \cdot b^t$
- Based on your regression model, what is the percent decrease per day?
- Using your model, find the number of nuclei after 5 days.
- Use your model to find the number of nuclei after 9 days. How does this compare to the data value in the table?
- Do the data values and regression values always match up? Why or why not?

For questions 22–24, start in part a) by creating a scatterplot. If the data is neither linear nor exponential, state that, and then do not continue to parts b) and c).

- 22) A large retail company wants to see if there is a relationship between the amount they spend on advertising, a , and their annual sales, S . The data is provided in the table below (both values are in millions of dollars).

$a = \text{amount spent on advertising}$	21	25	31	35	42	49	50	55
$S = \text{annual sales}$	601	635	886	1102	1199	1276	1407	1518

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine S when $a = 58$, and write your answer in a complete sentence. Round your answer to three decimal places.

- 23) You decide to start a YouTube channel, and are tracking the number of subscribers you see over the last several months. Below is the data showing the month, m (with month 0 representing the month you started the channel) and the total number of subscribers, S .

$m = \text{month}$	0	1	2	3	4	5	6
$S = \text{total subscribers}$	254	266	279	320	406	473	587

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine S when $m = 12$, and write your answer in a complete sentence. Round your answer to three decimal places.

24) A popular beach collected data on the daily high temperature (in degrees F) and the number of beachgoers for a sample of 10 days in May. The data is provided below:

$t = \text{temp}$	87	83	90	92	95	89	87	83	78	78
$N = \text{number of beachgoers}$	285	315	254	420	346	378	280	345	388	411

- Draw a scatterplot of the data. Based on your scatterplot, does the data appear to be linear, exponential, or neither?
- If the data appears either linear or exponential, use the appropriate regression to find the regression equation for the model. Round all values to 3 decimal places. Use the indicated variables.
- Determine S when $m = 12$, and write your answer in a complete sentence. Round your answer to three decimal places.

CHAPTER 5 – SIMPLE AND COMPOUND INTEREST

Chapter 5 Learning Objectives:

- 1) Recognize that simple interest grows linearly
- 2) Be able to correctly apply the simple interest formula to solve for A , r , t , or P
- 3) Recognize that compound interest grows exponentially
- 4) Correctly apply the compound interest formula to solve for A , r , or P
- 5) Recognize that the compound interest formula is used for solving when you have a one-time, lump sum deposit
- 6) Calculate the APY for an account earning compound interest

Sections

Section 5.1 – Simple Interest

Section 5.2 – Compound Interest

Section 5.3 – Annual Percentage Yield (APY)

SECTION 5.1: SIMPLE INTEREST

Discussing interest starts with the **principal**, or amount your account starts with initially. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year.

For most long term, simple interest loans, it is common for interest to be paid on an annual basis. In that case, interest would be earned each year on the principal.

For example, suppose you borrowed \$10,000 from a friend and agree to repay it with 3% annual interest, in 5 years. You would not only repay your friend the \$10,000 you borrowed. You would also pay simple interest for each year you had borrowed the money.

Year	Starting balance	Interest earned	Ending Balance
1	10,000.00	300.00	10,300.00
2	10,300.00	300.00	10,600.00
3	10,600.00	300.00	10,900.00
4	10,900.00	300.00	11,200.00
5	11,200.00	300.00	11,500.00

The total amount you would repay your friend would be \$11,500.00, which is the original principal plus the interest over 5 years. This process can be generalized with the formulas below.

Simple Interest over Time

$$I = Prt$$

$$A = P + I = P + Prt = P(1 + rt)$$

I is the dollar amount of interest

A is the balance in the account after t years

P is the balance in the account at the beginning (starting amount, or principal).

r is the annual interest rate (APR) in decimal form (Example: 5% = 0.05)

t is the number of years we plan to leave the money in the account

Bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value, along with any outstanding interest due.

Example 1: Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a \$5,000 bond that pays 4.5% simple interest annually, and matures in 5 years. Find the future value of the bond after 5 years.

Solution:

$$A = 5,000(1 + 0.045(5))$$

$$A = 5,000(1 + 0.225)$$

$$A = 5,000(1.225)$$

$$A = \$6,125$$

Answer: When the bond matures after 5 years, you would have received the \$5,000 you originally paid, plus \$1,125 in interest, for a total of \$6,125.

Section 5.1 You Try: Maria invests \$16,000 at 3% simple interest for 19 years.

- How much interest will she earn?
- How much is in the account at the end of the 19 year period?

Example 2: Martha decides to invest \$4500 into a savings account earning 2.5% simple interest. How long will it take her investment to double in value?

Solution: Since the account is earning 2.5% interest, $r = 0.025$. The initial amount invested is \$4,500, so $P = 4,500$

In this problem, we are not specifically given the future balance in the account, A , but we are told that we want to find the amount of time it will take for the initial investment to double in value.

Since the initial investment amount was \$4,500, we are looking for the value of t where $A = 2(\$4,500) = \$9,000$, which corresponds to the equation below.

$$9,000 = 4,500(1 + 0.025t)$$

$$2 = 1 + 0.025t$$

$$1 = 0.025t$$

$$t = 40$$

Answer: It will take 40 years for Martha's initial investment of \$4,500 to double in value.

SECTION 5.2: COMPOUND INTEREST

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**.

Suppose that we deposit \$1,000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR) – the total interest to be paid during the year.

Since our interest is being paid monthly, each month we will earn $\frac{1}{12}$ of the 3% annual interest,

or $\frac{3\%}{12} = 0.25\%$ per month.

In the first month,

$$P = \$1,000$$

$$r = 0.0025 \text{ (0.25\%)}$$

$$I = \$1,000 (0.0025) = \$2.50$$

$$A = \$1,000 + \$2.50 = \$1,002.50$$

In the first month, we will earn \$2.50 in interest, raising our account balance to \$1002.50.

In the second month,

$$P = \$1,002.50$$

$$I = \$1,002.50 (0.0025) = \$2.51 \text{ (rounded)}$$

$$A = \$1,002.50 + \$2.51 = \$1,005.01$$

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that **compounding** of interest gives us.

Calculating out the ending balance for the first 12 months of the starting balance (rounding to the nearest cent) yields the table below.

Month	Starting balance	Interest earned	Ending Balance
1	1000	2.5	1002.5
2	1002.5	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.1
7	1015.1	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

This process can be generalized with the formula below.

Compound Interest over Time

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

A is the balance in the account after t years

t is the number of years we plan to leave the money in the account

P is the balance in the account at the beginning (starting amount, or principal).

r is the annual interest rate (APR) in decimal form (Example: 5% = 0.05)

n is the number of compounding periods in one year.

Compounding Terminology

If the compounding is done annually (once a year), $n = 1$.

If the compounding is done quarterly, $n = 4$.

If the compounding is done monthly, $n = 12$.

If the compounding is done daily, $n = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account **once** and let it sit there earning interest.

Example 3: If you invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution: Since we are starting with \$3,000, $P_0 = 3,000$, our interest rate is 3%, so $r = 0.03$. Since we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for the ending value, A , when $t = 10$.

$$A = 3,000 \left(1 + \frac{0.03}{4} \right)^{4(10)} = \$4,045.05$$

Answer: The account will be worth \$4,045.05 in 10 years.

Section 5.2A You Try: If you invest \$3,000 in an investment account paying 3% interest compounded *weekly*, how much will the account be worth in 10 years?

Example 4: A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3,000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution: In this example,

$P = \$3,000$ the initial deposit

$r = 0.06$ 6% annual rate

$n = 12$ 12 months in 1 year

$t = 20$ after 20 years

$$A = 3000 \left(1 + \frac{0.06}{12} \right)^{12 \times 20} = \$9930.61 \quad (\text{round to the nearest cent})$$

Answer: You will have \$9930.61 in the account after 20 years.

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest of 6% per year versus 6% compounded monthly.

Years	5	10	15	20	25	30	35
Simple	\$3,900	\$4,800	\$5,700	\$6,600	\$7,500	\$8,400	\$9,300
Compound	\$4,046.55	\$5,458.19	\$7,362.28	\$9,930.61	\$13,394.91	\$18,067.73	\$24,370.65

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Example 5: A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to \$40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Solution: Since the account is earning 6%, $r = 0.06$. Since interest is compounded twice a year, $n = 2$

In this problem, we don't know how much we are starting with, so we will be solving for P_0 , the initial amount needed. We do know we want the end amount, A , to be \$40,000, so we will be looking for the value of P_0 so that $A = 40,000$.

$$40,000 = P \left(1 + \frac{0.06}{2} \right)^{2(18)}$$

$$40,000 = P(2.898278328)$$

$$P = \frac{40,000}{2.898278328} \approx \$13,801.30$$

Answer: Lily will need to invest \$13,801.30 to have \$40,000 in 18 years.

A Note on Rounding: It is important to be **very careful** about rounding when performing calculations. If possible, enter the entire calculation in one step into your calculator to avoid rounding error. If this is not possible, you want to keep as many decimals during calculations as you can. Try calculating example 5 using 2.898 instead of 2.898278328 and compare your answer with the example

Section 5.2B You Try: If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to \$40,000 over 18 years, and she believes the account will earn 6% compounded *daily*, how much will Lily need to invest in the account now?

SECTION 5.3: ANNUAL PERCENTAGE YIELD (APY)

Because of compounding throughout the year, with compound interest the actual increase in a year is usually *more* than the annual percentage rate. If \$1,000 were invested at 10%, the following table shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	\$1100
Semiannually	\$1102.50
Quarterly	\$1103.81
Monthly	\$1104.71
Daily	\$1105.16

If we were to compute the actual percentage increase for the daily compounding (last row of the table above), there was an increase of \$105.16 from an original amount of \$1,000, for a percentage increase of $\frac{105.16}{1000} = 0.10516 = 10.516\%$ increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year, $t = 1$, would be $A = P\left(1 + \frac{r}{n}\right)^n$.

To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount as shown below.

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$\frac{\text{Balance after one year} - \text{Starting Principal}}{\text{Starting Principal}} = \frac{P\left(1 + \frac{r}{n}\right)^n - P}{P} = \left(1 + \frac{r}{n}\right)^n - 1$$

or equivalently

$$APR = \left(1 + \frac{r}{n}\right)^n - 1 \quad (\text{convert to a percent})$$

The above formula should look very similar to the one we used to find the Relative Change in Chapter 4.

Example 6: Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

Solution: We can compare these rates using the annual percentage yield – the actual percent increase in a year.

$$\text{Bank A: } A = \left(1 + \frac{0.012}{4}\right)^4 - 1 = 0.01254 = 1.2054\%$$

$$\text{Bank B: } A = \left(1 + \frac{0.011}{12}\right)^{12} - 1 = 0.01056 = 1.1056\%$$

Answer: Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

Section 5.3 You Try: If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to \$40,000 over 18 years, and she believes the account will earn 6% compounded *daily*, how much will Lily need to invest in the account now?

- Calculate the APY for an account that pays 4% compounded daily.
- Calculate the APY for an account that pays 4% compounded monthly.
- Calculate the APY for an account that pays 4% compounded annually.

The table below shows the APY for \$1000 invested in an account that pays 10% at different compounding frequencies:

Frequency	Value after 1 year	APY
Annually	\$1100	10%
Semiannually	\$1102.50	10.25%
Quarterly	\$1103.81	10.381%
Monthly	\$1104.71	10.471%
Daily	\$1105.16	10.516%

Notice that the APY is always greater than the APR if interest is compounded more than once a year.

Chapter 5 – Answers to You Try Problems

5.1: At the end of the 19 year period, she will have earned \$9,120 in interest. The balance in her account will be \$25,120.

5.2A: The account will be worth \$4,049.23.

5.2B: Lily will need to invest \$13,585.03 to have \$40,000 in 18 years.

5.3:

- a) 4% compounded daily: $APY = 4.081\%$
- b) 4% compounded monthly: $APY = 4.074\%$
- c) 4% compounded annually: $APY = 4\%$

CHAPTER 5 – PRACTICE PROBLEMS

When needed, round your answer to the nearest cent.

- 1) Janelle invests \$18,000 at 8% simple interest for 41 years. How much is in the account at the end of the 41-year period?
- 2) Carlos invests \$15,000 at 2% simple interest for 29 years. How much is in the account at the end of the 29-year period?
- 3) Kristina invests \$13,000 at 7% simple interest for 32 years. How much is in the account at the end of the 32-year period?
- 4) How much would you need to deposit in an account now in order to have \$3,000 in the account in 5 years? Assume the account earns 2% simple interest.
- 5) How much would you need to deposit in an account now in order to have \$2,000 in the account in 15 years? Assume the account earns 6% simple interest.
- 6) You invest \$9600 in a savings account that pays 3.6% simple interest. How long will it take for this investment to double in value? Round your answer to the nearest tenth of a year.
- 7) Jenelle wants to invest \$500 in a savings account that pays 4.3% simple interest. How long will it take for this investment to triple in value? Round your answer to the nearest tenth of a year.
- 8) Brian invests \$1200 in a savings account that pays 5.1% simple interest. How long will it take for this investment to double in value? Round your answer to the nearest tenth of a year.
- 9) You deposit \$300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
- 10) How much will \$1,000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
- 11) You deposit \$9,400 in an account earning 4% interest compounded semi-annually. How much will you have in the account in 5 years?
- 12) You deposit \$2,100 in an account earning 2.5% interest compounded semi-annually. How much will you have in the account in 7 years?
- 13) You deposit \$4,000 in an account earning 8% interest compounded monthly. How much will you have in the account in 10 years?

College Math – Chapter 5 Practice Problems

- 14) You deposit \$200 in an account earning 5.2% interest compounded monthly. How much will you have in the account in 15 years?
- 15) You deposit \$2,000 in an account earning 3% interest compounded monthly.
- a) How much money will you have in the account in 20 years?
 - b) How much interest will you earn?
- 16) You deposit \$10,000 in an account earning 4% interest compounded monthly.
- a) How much money will you have in the account in 25 years?
 - b) How much interest will you earn?
- 17) You deposit \$5,300 in an account earning 5% interest compounded quarterly. How much will you have in the account in 6 years?
- 18) You deposit \$7,000 in an account earning 3.5% interest compounded quarterly. How much will you have in the account in 8 years?
- 19) You deposit \$3,000 in an account earning 8% interest compounded quarterly. How much will you have in the account in 10 years?
- 20) How much would you need to deposit in an account now in order to have \$4,000 in the account in 10 years? Assume the account earns 3% interest compounded monthly.
- 21) How much would you need to deposit in an account now in order to have \$2,500 in the account in 5 years? Assume the account earns 2.5% interest compounded monthly.
- 22) How much would you need to deposit in an account now in order to have \$6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
- 23) How much would you need to deposit in an account now in order to have \$20,000 in the account in 10 years? Assume the account earns 7% interest compounded quarterly.
- 24) How much would you need to deposit in an account now in order to have \$3,000 in the account in 15 years? Assume the account earns 2% interest compounded quarterly.
- 25) How much would you need to deposit in an account now in order to have \$7,000 in the account in 10 years? Assume the account earns 3% interest compounded annually.
- 26) You deposit \$23,000 in an account. Find the value of the investment at the end of 10 years if the account earns
- a) 7% interest compounded annually
 - b) 7% simple interest

- 27) You deposit \$4,000 in an account. Find the value of the investment at the end of 10 years if the account earns
- a) 5% interest compounded annually
 - b) 5% simple interest
- 28) A bank features a savings account that has an annual percentage rate of 2.8% with interest compounded monthly. Devin deposits \$9,500 into the account.
- a) How much money will Devin have in the account in 1 year?
 - b) What is the annual percentage yield (APY) for the savings account?
- 29) A bank features a savings account that has an annual percentage rate of 5.1% with interest compounded quarterly. Miranda deposits \$3,000 into the account.
- a) How much money will Miranda have in the account in 1 year?
 - b) What is the annual percentage yield (APY) for the savings account?
- 30) Calculate the APY for an account that pays 3.6% compounded monthly.
- 31) Calculate the APY for an account that pays 5.8% compounded semi—annually.
- 32) Calculate the APY for an account that pays 4.7% compounded quarterly.
- 33) Calculate the APY for an account that pays 2.3% compounded quarterly.
- 34) Calculate the APY for an account that pays 5.9% compounded annually.
- 35) Calculate the APY for an account that pays 6%
- a) compounded daily
 - b) compounded monthly
 - c) compounded annually
- 36) Calculate the APY for an account that pays 15%
- a) compounded daily
 - b) compounded monthly
 - c) compounded annually

CHAPTER 6 – ANNUITIES AND LOANS

Chapter 6 Learning Objectives:

- 1) Recognize that an annuity involves equal, regular payments or deposits
- 2) Be able to correctly apply the savings annuity formula and solve for any unknown using the TVM solver
- 3) Be able to correctly apply the payout annuity formula and solve for any unknown using the TVM solver
- 4) Be able to correctly apply the loan formula for amortized loans and solve for any unknown using the TVM solver
- 5) Solve real world problems that involve both planning for retirement and withdrawing from retirement accounts
- 6) Recognize how the proportions of a payment going toward principal and interest change over the life of a loan

Sections

Section 6.1 – Annuities

Section 6.2 – Payout Annuities

Section 6.3 – Loans

SECTION 6.1: ANNUITIES

Most of us aren't able to put a large sum of money in the bank today. Instead, we save by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. This is how most retirement accounts work.

In this course, we will assume that you put equal amounts of money into the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Suppose you start out by making monthly deposits of \$100 into an account that earns 12% annual interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. If payments are made monthly, we assume interest is compounded monthly. If payments are made annually, we assume interest is compounded annually. We will also assume in this course that all compounding and payments occur at the END of the month.

In this example with monthly payments, we assume the 12% is compounded monthly (or 1% each month). So, you make your first \$100 deposit at the end of the first month. At the end of the second month, you make another \$100 deposit PLUS you earn 1% interest on the prior balance (in this case 1% of \$100 = \$1) making the balance in your account \$201 at the end of month 2.

In general, **Ending Balance = Prior Balance + 1% Interest on Prior Balance + Monthly Payment**
The table below shows this process repeated for 6 months.

Month	Prior Balance	1% Interest on Prior Balance	Monthly Payment	Ending Balance
1	\$0	\$0	\$100	\$100
2	\$100	\$1	\$100	\$201
3	\$201	\$2.01	\$100	\$303.01
4	\$303.01	\$3.03	\$100	\$406.04
5	\$406.04	\$4.06	\$100	\$510.10
6	\$510.10	\$5.10	\$100	\$615.20

This process can be generalized by the Annuity Formula:

$$A = \frac{PMT \left(\left(1 + \frac{r}{n} \right)^{nt} - 1 \right)}{\left(\frac{r}{n} \right)}$$

A is the balance in the account after t years.

PMT is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate (APR) in decimal form (Example: 5% = 0.05)

n is the number of compounding periods in one year

t is the number of years we plan to make monthly deposits

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year. For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $n = 12$.

If you make your deposits every year, use yearly compounding, $n = 1$.

If you make your deposits every quarter, use quarterly compounding, $n = 4$.

Etc...

When do you use the Annuity formula?

Annuities assume that you put money in the account *on a regular schedule (every month, year, quarter, etc.)* and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: **One** deposit (Chapter 5)

Annuity: **Many** deposits (Chapter 6)

Example 1: A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If Charlotte deposits \$100 each month into an IRA earning 6% interest, how much will she have in the account after 20 years?

Solution: In this example,

$PMT = \$100$ the monthly deposit

$r = 0.06$ 6% annual rate

$n = 12$ since we're doing monthly deposits, we'll compound monthly

$t = 20$ we want the amount after 20 years

$$A = \frac{100 \left(\left(1 + \frac{0.06}{12} \right)^{12 \times 20} - 1 \right)}{\frac{0.06}{12}} = \$46,204.09$$

Answer: Charlotte's account will grow to \$46,204.09 after 20 years.

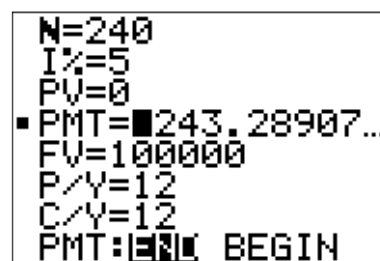
Notice that she deposited into the account a total of \$24,000 (\$100 a month for 240 months). The difference between what she ends up with and how much she put in is the **interest earned**. In this case it is $\$46,204.09 - \$24,000 = \$22,204.09$.

In the last example, the entire expression was entered into the calculator. This requires a lot of parentheses, including putting the fraction in the denominator in parentheses. This is so tedious, people created computer programs called TVM (Time Value of Money) to this more easily. The TVM Solver program on your graphing calculator can perform this calculation for you, and do it much quicker! The following guide will introduce you to the TVM solver.

Using the TVM Solver on your Graphing Calculator

Press **APPS**, then **Finance**, and **TVM Solver** (Note: If you have an older TI-83 calculator and do not have an APPS key, look for a key labeled Finance.)

You will see a screen like the following: (Note: You will see different numbers after each parameter. There's no reason to reset to zero--you will type right over them.)



- N** **Total number of payment/compounding periods** (this is $n \times t$ where n is the number of compoundings per year and t is the time in years)
- I%** **Annual interest rate (APR)** (Entered as a percent, **not** as a decimal.)
- PV** **Present Value** (the principal or beginning amount)
- PMT** **Payment Amount** (must be equal regular payments)
- FV** **Future Value** (accumulated value or end amount)
- P/Y** Number of payment periods per year
- C/Y** Number of compounding periods per year
- PMT:** For this class, keep the setting on END, indicating end-of-the month payments.

For our applications, C/Y will always be the same value as P/Y. If you change P/Y, your calculator will automatically reset C/Y.

NOTES:

- Cash outflows (paying money out) are considered negative. Enter them as negative values.
- Cash inflows (money coming in) are considered positive. Enter them as positive values.
- Deposits and loan payments are negative because they represent an amount you must pay *out* every month.

TVM Solver will calculate the value of **any** of the first five values in the above list (N, I%, PV, PMT or FV) if the other four values are known. To solve for an unknown value, you will put the cursor on the line you wish to find and then press SOLVE (press ALPHA, then the ENTER key).

Example 2: Repeat Example 1 with the TVM Solver

Solution: A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit \$100 each month into an IRA earning 6% interest. How much will you have in the account after 20 years?

We are looking for the future value (FV) of this account after 20 years. In this example,

$$N = 20 \times 12$$

$$I\% = 6$$

$$PV = 0 \text{ (There is no initial lump sum deposit)}$$

$$PMT = -100$$

FV = This is what you are solving for.

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, move your cursor to FV= and press ALPHA – ENTER.

$$FV = 46,204.08952$$

Answer: The future value (ending balance) will be \$46,204.09.

Example 3: You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution: In this example, we're looking for *PMT*.

$$r = 0.08$$

8% annual rate

$$n = 12$$

since we're depositing monthly

$$t = 30$$

30 years

$$A = \$200,000$$

The amount we want to have in 30 years

By hand, we could set up the equation, and solve for *d*:

$$200,000 = \frac{PMT \left(\left(1 + \frac{0.08}{12} \right)^{12 \times 30} - 1 \right)}{\frac{0.08}{12}} = \$46,204.09$$

$$PMT = \$134.20$$

Using the TVM Solver: We are looking for the monthly deposit required (PMT) to have a future value of \$200,000 in the account after 30 years, with an annual interest rate of 8%. In this example,

$$N = 30 \times 12$$

$$I\% = 8$$

$$PV = 0 \text{ (There is no initial lump sum deposit)}$$

PMT = This is what you are solving for.

$$FV = 200000$$

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, move your cursor to PMT = and press ALPHA – ENTER.

$$PMT = -134.20.$$

Answer: So you would need to deposit \$134.20 each month to have \$200,000 in 30 years if your account earns 8% interest.

Section 6.1A You Try: A more conservative investment account pays 3% interest. If you deposit \$5 a day into this account, how much will you have after 10 years? How much is from interest?

Example 4: You want to have \$200,000 in your account when you retire in 30 years. Your retirement account earns 8.2% interest. How much do you need to deposit each month to meet your retirement goal?

Solution: In this example,

$$N = 30 \times 12$$

$$I\% = 8.2$$

$$PV = 0 \text{ (There is no initial lump sum deposit)}$$

PMT = this is what you are solving for

$$FV = 200,000$$

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, move your cursor to PMT = and press ALPHA – ENTER.

$$PMT = -128.84224$$

Answer: You will need to deposit \$128.85 each month to reach the goal.

Section 6.1B You Try: If you want to end up with \$30,000 after 5 years in an account earning 5.2% interest, compounded monthly, how much is your monthly deposit?

Example 5: Emily invests \$250 a month for 5 years into an account earning 4% interest compounded monthly. After 5 years, she leaves the money, without making additional deposits, in the account for another 25 years. How much will Emily have in the end?

Solution: This problem requires two steps.

Step 1: Initially, monthly deposits are being made into a saving annuity for a period of 5 years. This is new for section 6.1.

Step 2: After that, the amount that has accrued from step 1 will sit (as a lump sum) and continue to earn compound interest for 25 more years. This is regular compound interest from section 5.2.

Step 1: Find the amount in the account after monthly payments are made for 5 years.

$$N = 12 \times 5$$

$$I\% = 4$$

$$PV = 0$$

$$PMT = -250$$

FV = this is what you are solving for

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, move your cursor to PMT = and press ALPHA – ENTER.

FV = \$16,574.74. After 5 years of making monthly payments, Emily will have \$16,574.74 in the account.

Step 2: Find the amount in the account after 25 years.

Use the compound interest formula from the previous chapter.

$$A = P \left(1 + \frac{r}{n} \right)^{n \times t} = 16,574.74 \left(1 + \frac{0.04}{12} \right)^{12 \times 25} = \$44,979.95$$

Answer: The final amount Emily will have in the account is \$44,979.95

SECTION 6.2: PAYOUT ANNUITIES

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account. In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty. Payout annuities are typically used after retirement. Perhaps you have saved \$500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity.

Payout Annuity Formula

$$P_0 = \frac{PMT \left(1 - \left(1 + \frac{r}{n} \right)^{-nt} \right)}{\frac{r}{n}}$$

P_0 is the balance in the account at the beginning (starting amount, or principal).

PMT is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (APR) in decimal form (Example: 5% = 0.05)

n is the number of compounding periods in one year.

t is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use the Payout Annuity Formula?

Payout annuities assume that you **take** money from the account **on a regular schedule (every month, year, quarter, etc.)** and let the rest sit there earning interest.

Compound interest: **One** deposit

Annuity: **Many** deposits.

Payout Annuity: **Many withdrawals**

Example 6: After retiring, you want to be able to take \$1,000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

Solution: Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.06}{12} \right)^{-12 \times 20} \right)}{\frac{0.06}{12}} = \$139,580.77$$

Using the TVM Solver:

$$N = 12 \times 20$$

$$I\% = 6$$

PV = this is what you are solving for

$$PMT = 1000$$

$$FV = 0$$

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, solve for PV. $PV = -139,580.7717$

Answer: You will need \$139,580.78 in the account by the time you retire.

Example 7: You know you will have \$500,000 in your account when you retire. You want to be able to take annual withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each year?

Solution: Putting this into the equation:

$$P_0 = \frac{1000 \left(1 - \left(1 + \frac{0.08}{1} \right)^{-1 \times 30} \right)}{\frac{0.08}{1}} = \$44,413.72$$

Using the TVM Solver:

$$N = 1 \times 30$$

$$I\% = 8$$

$$PV = -500,000$$

PMT = this is what you are solving for

$$FV = 0$$

$$P/Y = 1$$

$$C/Y = 1$$

PMT: END

Once all of the given parameters are entered in, solve for PMT. $PMT = 44,413.71669$

Answer: So you would be able to withdraw \$44,413.72 each year for 30 years.

Section 6.2 You Try: After retiring, you want to be able to take \$50,000 every year for a total of 25 years from your retirement account. The account earns 7% interest. How much will you need in your account when you retire?

Example 8: You want to be able to withdraw \$65,000 from your account each year for 30 years after you retire. You expect to retire in 25 years. If your account earns 6% interest, how much will you need to deposit each year until retirement to achieve your retirement goals?

Solution: This problem requires two steps.

Step 1: Initially, figure out how much money you need to have in your account at retirement so that you can withdraw \$65,000 per year for 30 years.

Step 2: After that, figure out how much money you need to deposit every year before retirement, so that you have the amount needed from Step 1 after 25 years.

Step 1: Find the amount you need to have in your account at retirement so that you can make annual withdrawals of \$65,000 per year for 30 years.

$$N = 1 \times 30$$

$$I\% = 6$$

PV = this is what you are solving for

$$PMT = -65000$$

$$FV = 0$$

$$P/Y = 1$$

$$C/Y = 1$$

PMT: END

Once all of the given parameters are entered in, move your cursor to PV= and press ALPHA – ENTER.

PV = \$894,714.02. You need to have \$894,714.02 in your retirement account so that you can withdraw \$65,000 per year for 30 years.

Step 2: Find the amount you need to deposit each year in order to have \$894,714.02 in 25 years.

$$N = 1 \times 25$$

$$I\% = 6$$

$$PV = 0$$

PMT = this is what you are solving for

$$FV = 894,714.02$$

$$P/Y = 1$$

$$C/Y = 1$$

PMT: END

Once all of the given parameters are entered in, move your cursor to PMT= and press ALPHA – ENTER.

PMT = -\$16,307.70.

Answer: You need to deposit \$16,307.70 each year for 25 years, so that you will have enough money in your account when you retire to withdraw \$65,000 per year for 30 years.

SECTION 6.3: LOANS

In the last section, you learned about payout annuities. In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had \$10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests \$10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

Suppose you borrow \$1,000 at a 12% annual interest rate. Again, we assume that the account is compounded with the same frequency as we make payments unless stated otherwise.

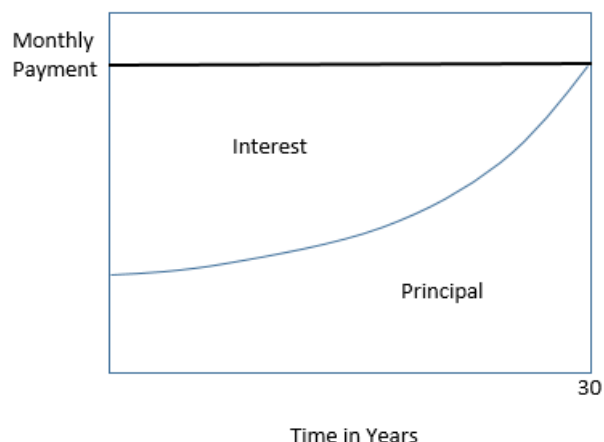
In this example with monthly payments, we assume the 12% is compounded monthly (or 1% each month). So, you make your first \$100 payment at the end of the first month. At the end of the second month, you make another \$100 payment, but you also have to add 1% interest on the prior balance (in this case 1% of \$1,000 = \$10) making the new balance (amount owed) \$910 at the end of month 2.

In general, **Ending Balance = Prior Balance + 1% Interest on Prior Balance – Monthly Payment**

The table below shows this process over 4 months.

Month	Prior Balance	1% Interest on Prior Balance	Monthly Payment	Ending Balance
1				\$1,000
2	\$1,000	\$10	\$100	\$910
3	\$910	\$9.10	\$100	\$819.10
4	\$819.10	\$8.19	\$100	\$727.29
And so on	...	...	...	...

Notice that each month, the interest that must be paid decreases slightly (because you have made a payment that has decreased the principal). In an installment loan, the monthly payments are a fixed amount, so early on in the loan a larger proportion of your payment is being applied to interest, and a smaller proportion of your payment is being applied toward paying down the principal. ***As the term of the loan proceeds, the proportion of the payment going toward interest gradually decreases, and the proportion of the payment going toward principal gradually increases.***



Example 9: You take out a 5 year installment loan to purchase a new vehicle. Your monthly payment is fixed at \$485.22 for those 5 years, after which you will have paid off the loan in full. Will a larger proportion of your payment go towards principal during year 1 of the loan, or year 4 of the loan?

Solution: You owe the most money on your car when you first buy it. Since the interest due is calculated based on how much you currently owe on the loan, you owe the most interest during the first month. Keep in mind that your payment is a fixed amount, \$485.22. Early in the loan, a larger proportion of your payment is going towards paying interest, while a smaller proportion of your payment is being applied to principal. As you gradually decrease your principal (the amount you owe), your interest due each month decreases as well. This means that over the life of the loan, a larger proportion of your payment can be used to pay down the principal.

Answer: During year 4 of the loan, a larger proportion of the fixed payment will go towards principal than during year 1 of the loan.

The following formula generalizes the installment loan process:

Loan Formula

$$P_0 = \frac{PMT \left(1 - \left(1 + \frac{r}{n} \right)^{-nt} \right)}{\frac{r}{n}}$$

P is the balance in the account at the beginning (starting amount, or principal).

PMT is your loan payment (your monthly payment, annual payment, etc)

r is the annual interest rate (APR) in decimal form (Example 5% = 0.05)

n is the number of compounding periods in one year.

t is the number of years we plan to borrow the money (length of the loan, in years)

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use the Loan Formula?

The loan formula assumes that you make loan payments on a regular schedule (every month, year, quarter, etc.) and are paying interest on the loan.

Compound interest: **One** deposit

Annuity: **Many** deposits.

Payout Annuity: **Many withdrawals**

Loans: **Many payments**

Example 10: Abigale can afford \$200 per month as a car payment. If she can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can she afford (in other words, what amount loan can Abigale pay off with \$200 per month)?

Solution: In this example, we are looking for P_0 , the starting amount of the loan.

Putting this into the equation:

$$P_0 = \frac{200 \left(1 - \left(1 + \frac{0.03}{12} \right)^{-5 \times 12} \right)}{\frac{0.03}{12}} = \$11,130.47$$

Answer: So Abigale can afford an \$11,130.47 loan.

Note: Abigail will pay a total of \$12,000 (\$200 per month for 60 months) to the loan company. The difference between the amount she pays and the amount of the loan is the **interest paid**. In this case, Abigale is paying $\$12,000 - \$11,130.47 = \$869.53$ interest total.

You can also use TVM Solver to do this calculation.

Example 10b: Repeat Previous Example Using TVM Solver

Abigale can afford \$200 per month as a car payment. If she can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can she afford (in other words, what amount loan can Abigale pay off with \$200 per month)?

Solution: In this example, we are looking for PV , the starting amount of the loan.

$$N = 12 \cdot 5$$

$$I\% = 3$$

PV = this is what you are solving for

$$PMT = -1200$$

$$FV = 0$$

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, solve for PV to get $PV = 11,130.47153$

Answer: So Abigale can afford a car loan of up to \$11,130.47.

Example 11: You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution: Use the TVM Solver

$$N = 12 \cdot 30$$

$$I\% = 6$$

$$PV = 140,000$$

PMT = this is what you are solving for

$$FV = 0$$

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, solve for PMT to get $PMT = -839.37073$

Answer: So you will make payments of \$839.38 per month for 30 years.

Note: You're paying a total of \$302,176.80 to the loan company (\$839.38 per month for 360 months). You are paying a total of $\$302,176.80 - \$140,000 = \$162,176.80$ in interest over the life of the loan.

Section 6.3A You Try: Janine bought \$3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

With loans, it is often desirable to determine what the **remaining loan balance** will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were \$1,000 a month, after a year you will *not* have paid off \$12,000 of the loan balance.

Example 12: (Continued from Example 11)

You want to take out a \$140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. Your monthly payments are \$839.38. How much will still be owed after making payments for 10 years?

Solution: Use the TVM Solver

$$N = 12 \cdot 10$$

$$I\% = 6$$

$$PV = 140,000$$

$$PMT = -839.38$$

FV = this is what you are solving for

$$P/Y = 12$$

$$C/Y = 12$$

PMT: END

Once all of the given parameters are entered in, solve for FV to get $FV = -117,158.50$

Answer: So you will still owe \$117,158.50 after making payments for 10 years.

Section 6.3B You Try: You finance a \$30,000 car at 2.9% interest for 60 months (5 years). What will your monthly payments be? How much will you still owe on the car after making payments for 2 years?

NOTE: Be sure to round the monthly payment UP to the nearest cent. Then use this (rounded) value when calculating the amount still owed. Round the amount owed UP to the nearest cent.

Chapter 6 – Answers to You Try Problems

6.1A: After 10 years, you will have \$21,282.07 in the account. \$3,032.07 is from interest.

6.1B: Your monthly deposit should be \$438.89.

6.2: You will need \$582,679.16 in your account when you retire.

6.3A: She will have to pay \$146.89 each month.

6.3B: Monthly payments: \$537.73 per month
You will still owe \$18,518.61 after 2 years.

CHAPTER 6 – PRACTICE PROBLEMS

When needed, round your answer to the nearest cent.

- 1) You deposit \$1,000 each year into an account earning 4% interest compounded annually. How much will you have in the account in 25 years?
- 2) You deposit \$3,000 each year into an account earning 7% interest compounded annually. How much will you have in the account in 15 years?
- 3) You deposit \$1,000 each year into an account earning 8% compounded annually.
 - a) How much will you have in the account in 10 years?
 - b) How much total money will you put into the account?
 - c) How much total interest will you earn?
- 4) You deposit \$100 each month into an account earning 6% interest compounded monthly.
 - a) How much will you have in the account in 15 years?
 - b) How much total money will you put into the account?
 - c) How much total interest will you earn?
- 5) You deposit \$200 each month into an account earning 3% interest compounded monthly.
 - a) How much will you have in the account in 30 years?
 - b) How much total money will you put into the account?
 - c) How much total interest will you earn?
- 6) Suppose you want to have \$500,000 for retirement in 20 years. Your account earns 10% interest.
 - d) How much would you need to deposit in the account each month?
 - e) How much total money will you put into the account?
 - f) How much interest will you earn?
- 7) Jose has determined he needs to have \$800,000 for retirement in 30 years. His account earns 6% interest.
 - g) How much would you need to deposit in the account each month?
 - h) How much total money will you put into the account?
 - i) How much total interest will you earn?
- 8) You wish to have \$3,000 in 2 years to buy a new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
- 9) After retiring, you want to be able to withdraw \$1,500 each month for 25 years. Your account earns 6% interest. How much do you need in your account when you retire?

- 10) After retiring, you want to be able to withdraw \$20,000 each year for 15 years. Your account earns 7% interest. How much do you need in your account when you retire?
- 11) How much money will you need to have at retirement so you can withdraw \$60,000 a year for 20 years from an account earning 8% compounded annually?
- 12) You want to be able to withdraw \$30,000 each year for 25 years. Your account earns 8% interest.
- How much do you need in your account at the beginning?
 - How much total money will you pull out of the account?
 - How much of that money is interest?
- 13) You want to be able to withdraw \$35,000 each year for 20 years. Your account earns 6% interest.
- How much do you need in your account at the beginning?
 - How much total money will you pull out of the account?
 - How much of that money is interest?
- 14) You have \$500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 20 years?
- 15) You have \$400,000 saved for retirement. Your account earns 7% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 25 years?
- 16) You have \$300,000 saved for retirement. Your account earns 9% interest. How much will you be able to pull out each month, if you want to be able to take withdrawals for 15 years?
- 17) You can afford a \$700 per month mortgage payment. You can get a loan at 5% interest for 30 years.
- What is the largest loan you can afford?
 - How much total money will you pay the loan company?
 - How much of that money is interest?
- 18) Marie can afford a \$250 per month car payment. She can get a loan at 7% interest for 5 years.
- How expensive of a car can she afford?
 - How much total money will she pay the loan company?
 - How much of that money is interest?

College Math – Chapter 6 Practice Problems

- 19) You can afford a \$1,500 per month mortgage payment. You can get a loan at 8% interest for 25 years.
- How big of a loan can you afford?
 - How much total money will you pay the loan company?
 - How much of that money is interest?
- 20) You can afford an \$850 per month mortgage payment. You can get a loan at 6% interest for 30 years.
- What is the largest loan you afford?
 - How much total money will you pay the loan company?
 - How much of that money is interest?
- 21) You want to buy a \$28,000 car. You can make a 10% down payment, and will finance the balance with a 2% interest rate for 48 months (4 years). What will your monthly payments be?
- 22) You want to buy a \$32,000 car. You can make a 10% down payment, and will finance the balance with a 5% interest rate for 36 months (3 years). What will your monthly payments be?
- 23) You want to buy a \$200,000 home. You plan to pay 10% as a down payment, and take out a 30-year loan for the rest.
- How much is the loan amount going to be?
 - What will your monthly payments be if the interest rate is 5%?
 - What will your monthly payments be if the interest rate is 6%?
- 24) Lynn bought a \$300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
- Find her monthly payments.
 - How much interest will she pay over the life of the loan?
- 25) You want to buy a \$25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
- 26) You want to buy a \$33,000 car. The company is offering a 4% interest rate for 36 months (3 years). What will your monthly payments be?
- 27) You want to buy a \$16,000 car. The company is offering a 3% interest rate for 48 months (4 years). What will your monthly payments be?
- 28) You decide finance a \$12,000 car at 3% compounded monthly for 4 years.
- What will your monthly payments be?
 - How much interest will you pay over the life of the loan?

- 29) Emily bought a car for \$24,000 three years ago. The loan had a 5-year term at 3% interest rate. How much does she still owe on the car?
- 30) A friend bought a house 15 years ago, taking out a \$120,000 mortgage at 6% for 30 years. How much does she still owe on the mortgage?
- 31) Suppose you invest \$140 a month for 6 years into an account earning 7% compounded monthly. After 6 years, you leave the money, without making additional deposits, in the account for another 26 years. How much will you have in the end?
- 32) Suppose you invest \$170 a month for 5 years into an account earning 6% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 29 years. How much will you have in the end?
- 33) Lucy invests \$130 a month for 7 years into an account earning 9% compounded monthly. After 7 years, she leaves the money, without making additional deposits, in the account for another 23 years. Alternatively, Monica didn't invest anything for the first 7 years, then deposited \$130 a month for 23 years into an account earning 9% compounded monthly.
- a) How much total money will Lucy have in the end?
 - b) How much total money will Monica have in the end?
 - c) How much of the total amount in Lucy's account is interest?
 - d) How much of the total amount in Monica's account is interest?
 - e) Which account had more total amount of money? Which account made more interest? Based on these observations, who made the better investment?

CHAPTER 7 – SETS AND VENN DIAGRAMS

Chapter 7 Learning Objectives:

- 1) Use correct set notation and terminology
- 2) Find intersection, unions, and complements of sets
- 3) Model relationships between sets using Venn diagrams
- 4) Recognize and be able to apply cardinality notation
- 5) Use cardinality notation and formulas to solve Venn diagrams

Sections

Section 7.1 – Set Basics

Section 7.2 - Union, Intersection, and Complement

Section 7.3 - Venn Diagrams

Section 7.4 - Cardinality

SECTION 7.1: SET BASICS

It is natural for us to classify items into groups, or sets, and consider how those sets overlap with each other. We can use these sets to understand relationships between groups, and to analyze survey data.

An art collector might own a collection of paintings, while a music lover might keep a collection of CDs. Any collection of items can form a **set**.

Definition: A **set** is a collection of distinct objects, called **elements** of the set.

A set can be defined by describing the contents, or by listing the elements of the set, enclosed in curly brackets.

Example 1: Some examples of sets are defined by describing the contents of the set:

- a) The set of all even numbers
- b) The set of all books written about travel to Chile

Other sets are defined by listing the elements of the set:

- a) {1, 3, 9, 12}
- b) {red, orange, yellow, green, blue, indigo, purple}

A set simply specifies the contents; order is not important. The set represented by {1, 2, 3} is equivalent to the set {3, 1, 2}, however for convention, sets containing numbers usually list the elements from smallest to largest.

Notation: We will use a variable (S , U , etc.) to represent a set, to make it easier to refer to that set later. The symbol $\in$ means “is an element of”.

Definition: A set that contains no elements, $\{ \}$, is called the **empty set** and is written $\emptyset$.

Example 2: Let $A = \{1, 2, 3, 4\}$

To notate that 2 is an element of the set, we’d write $2 \in A$

Sometimes a collection might not contain all the elements of a set. For example, Chris owns three Madonna albums. While Chris’s collection is a set, we can also say it is a **subset** of the larger set of all Madonna albums.

Definitions: A **subset** of a set A is another set that contains only elements from the set A , but might not contain all the elements of A . If B is a subset of A , we write $B \subseteq A$

A **proper subset** is a subset that is not identical to the original set – it contains fewer elements. If B is a proper subset of A , we write $B \subset A$

Example 3: Consider the three sets below.

$$A = \text{the set of all even numbers} \quad B = \{2, 4, 6\} \quad C = \{2, 3, 4, 6\}$$

Here $B \subset A$ since every element of B is also an even number, so is an element of A . It is also true that $B \subset C$. C is not a subset of A , since C contains an element, 3, that is not contained in A .

Example 4: Suppose a set contains the plays “Much Ado About Nothing”, “MacBeth”, and “A Midsummer’s Night Dream”. What is a larger set this might be a subset of?

Solution: There are many possible answers here. One would be the set of plays by Shakespeare. This is also a subset of the set of all plays ever written. It is also a subset of all British literature.

Section 7.1 You Try: The set $A = \{1, 3, 5\}$. What is a larger set that this might be a subset of?

SECTION 7.2: UNION, INTERSECTION, AND COMPLEMENT

It is very common for sets to interact. For example, you and a new roommate decide to have a house party, and you both invite your circle of friends. At this party, two sets are being combined, though it might turn out that there are some friends that were in both sets.

Definitions: The **union** of two sets contains all the elements contained in either set A , or set B , or in both sets. The union is written “ $A \cup B$ ” or “ A or B ”.

The **intersection** of two sets contains only the elements that are in both sets A and B . The intersection is written “ $A \cap B$ ” or “ A and B ”.

The **complement** of a set A contains everything that is *not* in the set A . The complement of A can be written A^c , $\overline{A}$, A' not A , or sometimes $\sim A$. For consistency, in this text we will use A' .

Example 5: Consider the sets:

$$A = \{\text{red, green, blue}\}$$

$$B = \{\text{red, yellow, orange}\}$$

$$C = \{\text{red, orange, yellow, green, blue, purple}\}$$

$$D = \{\text{yellow, white}\}$$

a) Find $A \cup B$

b) Find $A \cap B$

c) Find $A' \cap C$

d) Find $A \cap D$.

Solutions:

a) The union contains all the elements in either set A or set B, or both:

$$A \cup B = \{\text{red, green, blue, yellow, orange}\}$$

(Notice we only list red once.)

b) The intersection contains all the elements in both sets: $A \cap B = \{\text{red}\}$

c) Here we're looking for all the elements that are **not** in set A and are also in C.

$$A' \cap C = \{\text{orange, yellow, purple}\}$$

d) Because set A and set B have no elements in common, we can write:

$$A \cap D = \emptyset \quad \text{or} \quad A \cap D = \{ \}$$

Section 7.2A You Try: Using the sets from the previous example, find $A \cup C$ and $B' \cap A$

Notice that in the example above, it would be hard to just ask for A' since everything from the color fuchsia to puppies and peanut butter are included in the complement of the set. For this reason, complements are usually only used with intersections, or when we have a universal set in place.

Definition: A **universal set** is a set that contains all the elements we are interested in.

This would have to be defined by the context. A complement is relative to the universal set, so A' contains all the elements in the universal set that are not in A.

Example 6: Examples of Universal Sets

- If we were discussing searching for books, the universal set might be all the books in the library.
- If we were grouping your Facebook friends, the universal set would be all your Facebook friends.
- If you were working with sets of numbers, the universal set might be all whole numbers, all integers, or all real numbers.
- If you were working with letters, the universal set might include all 26 letters in the English alphabet.

Example 7: Suppose the universal set is U = all whole numbers from 1 to 9, and $A = \{1, 2, 4\}$. Find A' .

Answer: $A' = \{3, 5, 6, 7, 8, 9\}$

Section 7.2B You Try: Suppose the universal set is U = all odd numbers from 1 to 15.

If $A = \{1, 3, 9, 11\}$, then $A' =$ _____

Set Operations

As we saw earlier with the expression $A' \cap C$ operations can be grouped together. Grouping symbols can be used like they are with arithmetic – to force an order of operations.

Example 8: Suppose the universal set $U = \{\text{cat, dog, rabbit, mouse, cow, duck, pig, deer, frog}\}$. Also define the following sets: $H = \{\text{cat, dog, rabbit, mouse}\}$, $F = \{\text{dog, cow, duck, pig, rabbit}\}$, and $W = \{\text{duck, rabbit, deer, frog, mouse}\}$.

Find the following: a) $(H \cap F) \cup W$ b) $H \cap (F \cup W)$ c) $(H \cap F)' \cap W$

Solutions:

a) Find $(H \cap F) \cup W$

We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$

Now we union that result with W :

Answer: $(H \cap F) \cup W = \{\text{dog, duck, rabbit, deer, frog, mouse}\}$

b) Find $H \cap (F \cup W)$

We start with the union:

$$F \cup W = \{\text{dog, cow, rabbit, duck, pig, deer, frog, mouse}\}$$

Now we intersect that result with H :

Answer: $H \cap (F \cup W) = \{\text{dog, rabbit, mouse}\}$

c) $(H \cap F)' \cap W$

We start with the intersection: $H \cap F = \{\text{dog, rabbit}\}$

Next we find: $(H \cap F)' \cap W = \{\text{cat, mouse, cow, duck, pig, deer, frog}\}$

Note: these are all elements that are in the universal set, but NOT in set $H \cap F$.

Now we want to find the elements that are not in $H \cap F$ and are also in W . This will be elements that are in the set above AND in set W :

Answer: $(H \cap F)' \cap W = \{\text{duck, deer, frog, mouse}\}$

SECTION 7.3: VENN DIAGRAMS

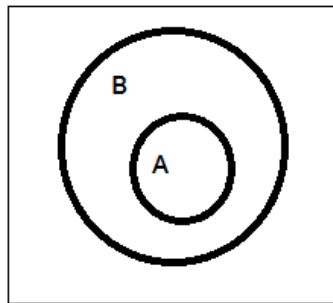
To visualize the interaction of sets, John Venn in 1880 thought to use overlapping circles, building on a similar idea used by Leonhard Euler in the 18th century. These illustrations are now called **Venn Diagrams**.

Definition: A Venn diagram represents each set by a circle, usually drawn inside of a rectangle. The rectangle represents the universal set. Overlapping areas of circles indicate elements common to both sets. (Note: There is no meaning to the size of the circle.) Basic Venn diagrams can illustrate the interaction of two or three sets

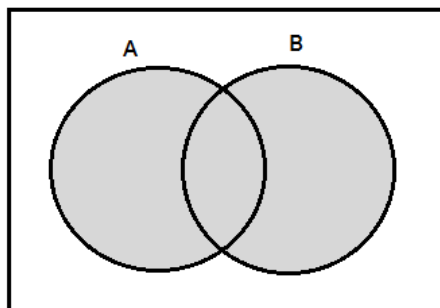
Example 9: Create Venn diagrams to illustrate a) $B \subset A$, b) $A \cup B$, c) $A \cap B$, and d) $A' \cap B$

Solutions:

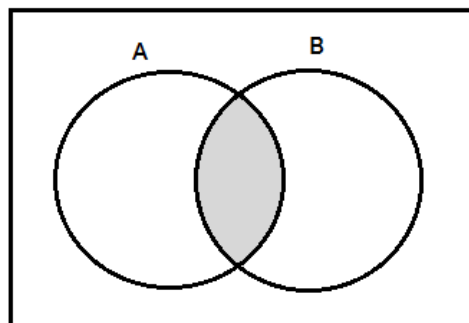
a) $A \subset B$ means that A is a proper subset of set B. So all elements in set A are also elements in Set B.



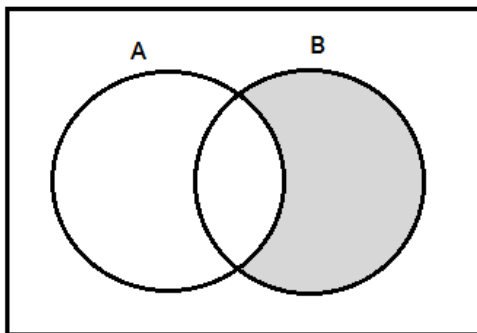
b) $A \cup B$ contains all elements in *either* set A, or set B, or both.



c) $A \cap B$ contains only those elements in both sets – in the overlap of the circles.



d) A' will contain all elements *not* in the set A . $A' \cap B$ will contain the elements in set B that are not in set A .



Section 7.3 You Try: Draw a Venn Diagram to represent that set G is a proper subset of set P .

SECTION 7.4: CARDINALITY

Often times we are interested in the number of items in a set or subset. This is called the cardinality of the set.

Definition: The number of elements in a set is the cardinality of that set. The cardinality of the set A is often notated as $n(A)$.

Example 10: Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8\}$.

Find the following: a) $n(B)$ b) $n(A \cup B)$ c) $n(A \cap B)$

Solution:

a) $n(B) = 4$, since there are 4 elements in the set.

b) Since $A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$, which contains 7 elements, $n(A \cup B) = 7$.

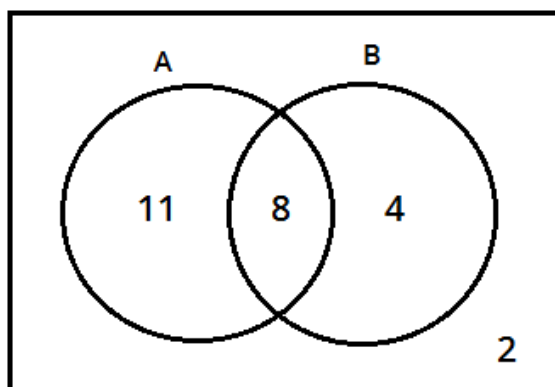
c) Since $A \cap B = \{2, 4, 6\}$, which contains 3 elements, $n(A \cap B) = 3$

Example 11: Find $n(P)$ where P = the set of English names for the months of the year?

Solution: $n(P) = 12$, since there are 12 months in the year.

Example 12: Use the Venn Diagram below to determine the following:

- a) $n(A)$ b) $n(A')$ c) $n(B)$ d) $n(B')$ e) $n(A \cup B)$ f) $n(A \cap B')$

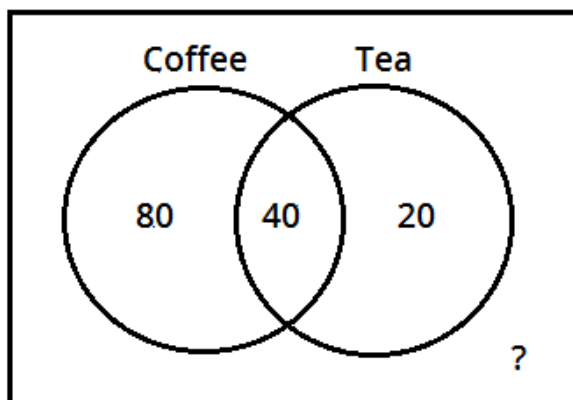


- Solutions:**
- | | |
|------------------------------------|--------------------------|
| a) $n(A) = 11 + 8 = 19$ | b) $n(A') = 4 + 2 = 6$ |
| c) $n(B) = 8 + 4 = 12$ | d) $n(B') = 11 + 2 = 13$ |
| e) $n(A \cup B) = 11 + 8 + 4 = 23$ | f) $n(A \cap B') = 11$ |

Sometimes (as in the previous example) we may be interested in the cardinality of the union or intersection of sets, but not know the actual elements of each set. This is common in surveying applications like the ones below.

Example 13: A survey asks 200 people “What beverage do you drink in the morning?”, and offers choices: Tea only, Coffee only, both coffee and tea. Suppose 20 report tea only, 80 report coffee only, 40 report both. How many people drink tea in the morning? How many people drink neither tea nor coffee?

Solution: This question can most easily be answered by creating a Venn diagram. We can see that we can find the people who drink tea by adding those who drink only tea to those who drink both coffee and tea: 60 people.



We can also see that those who drink neither are those not contained in any of the three other groupings, so we can count those by subtracting from the cardinality of the universal set, 200.

Answer: $200 - 20 - 80 - 40 = 60$ people who drink neither.

Example 14A: A survey asks: Which online services have you used in the last month: Twitter, Facebook, have used both. The results show 42% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. What percentage of people have used neither Twitter nor Facebook?

Solution: Let T be the set of all people who have used Twitter, and F be the set of all people who have used Facebook. Notice that while the cardinality of F is 70% and the cardinality of T is 42%, the cardinality of $F \cup T$ is not simply $70\% + 42\%$, since that would count those who use both services twice (they were added in as Twitter users, and added in again as Facebook users). To find the cardinality of $F \cup T$, (the number of people who used Facebook or Twitter) we can add the cardinality of F and the cardinality of T , then subtract out those in intersection that we've counted twice (those who used both Facebook and Twitter). In symbols,

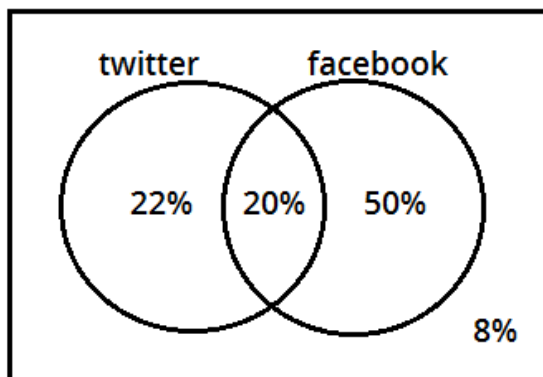
$$n(F \cup T) = n(F) + n(T) - n(F \cap T)$$

$$n(F \cup T) = 70\% + 42\% - 20\% = 92\%$$

We now know that 92% of those surveyed used Facebook, or Twitter, or both.

Now, to find how many people have not used either service, we're looking for the cardinality of $(F \cup T)'$. Since the universal set contains 100% of people and the cardinality of $F \cup T = 92\%$, the cardinality of $(F \cup T)'$ must be the other 8%.

The Venn diagram for this situation would look like this:



The previous example illustrated two important properties:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A') = n(U) - n(A)$$

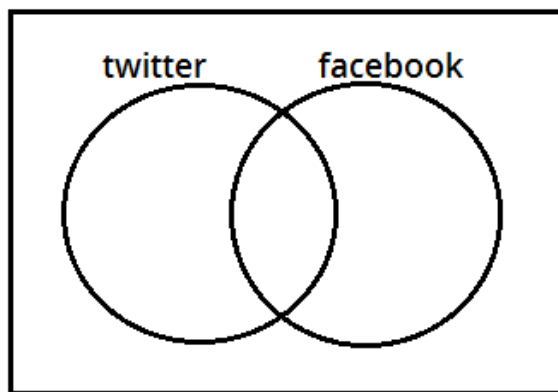
Notice that the first property can also be written in an equivalent form by solving for the cardinality of the intersection:

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

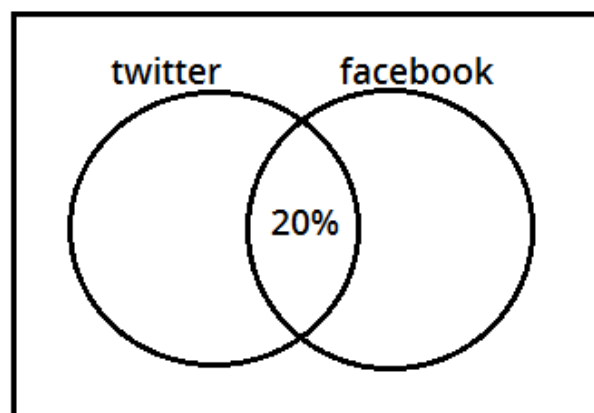
We can also use another technique for drawing Venn diagrams to model real life data. Consider the previous example again:

Example 14B: A survey asks: Which online services have you used in the last month: Twitter, Facebook, have used both. The results show 42% of those surveyed have used Twitter, 70% have used Facebook, and 20% have used both. What percentage of people have used neither Twitter nor Facebook?

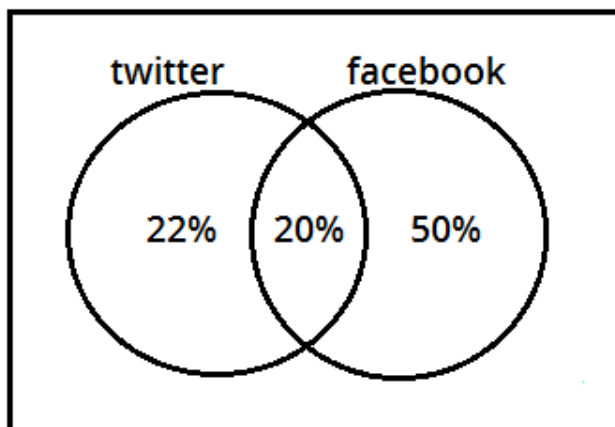
Solution: We solved this problem previously using the cardinality formulas, but some students appreciate a more visual approach which is what will be presented here. Before specifically looking at what question we are trying to answer, we will just work through drawing the Venn diagram to model the situation. We know the basic structure will look like this:



When modeling with a Venn diagram, ***always start in the middle*** (in other words, the intersection of your sets). We know that the portion of the diagram where the two circles overlap represents the intersection of the two sets (people who used both Twitter and Facebook). In this case, that is 20%. We start by labeling that part of the Venn diagram.

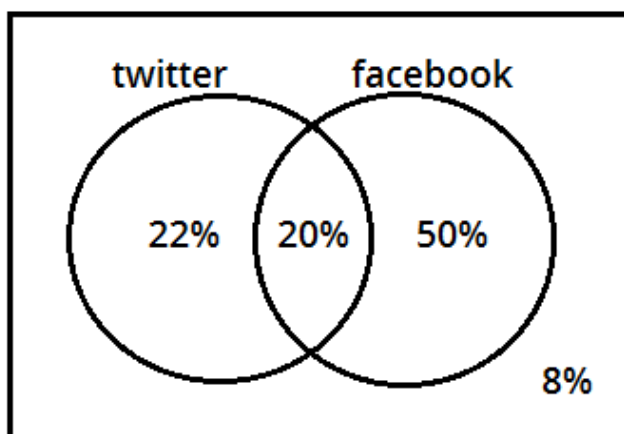


We now start working our way out. We know that the **entire** Twitter circle contains 42% of those surveyed, but we already know where 20% of them are (in the intersection that we just labeled). That leaves $42\% - 20\% = 22\%$ who used Twitter **only**. Using the same logic, of the 70% who used Facebook we know 20% used both Facebook and Twitter. That leaves $70\% - 20\% = 50\%$ who used Facebook **only**. The updated Venn diagram now looks like this:



Finally, since the universal set contains 100% of people surveyed, the percentage who used neither Twitter nor Facebook must be $100\% - 22\% - 20\% - 50\% = 8\%$.

The final Venn diagram for this situation would look like this:



Answer: 8% of people have used neither Twitter nor Facebook.

Section 7.4 You Try: In a survey of 115 pet owners, 26 said they own a cat, and 64 said they own a dog. 5 said they own both a dog and a cat. Use a Venn diagram to determine how many of the pet owners surveyed owned neither a cat nor a dog?

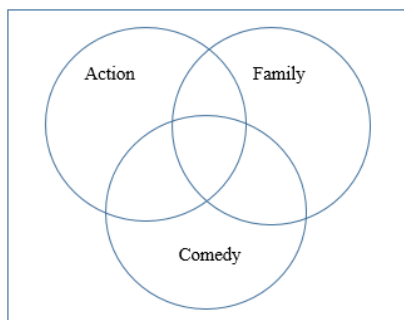
Consider the following additional examples of using Venn diagrams to model real life scenarios with three sets, instead of two.

Example 15: A group of adults was surveyed about the types of movies they enjoyed watching. The results were as follows:

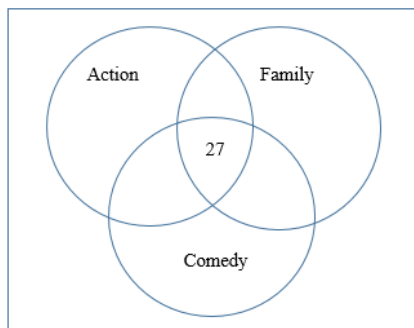
- 30 enjoyed watching only Action movies
- 46 enjoyed watching only Family movies
- 42 enjoyed watching only Comedy movies
- 57 enjoyed watching only Action and Family movies
- 82 enjoyed watching only Action and Comedy movies
- 23 enjoyed watching only Family and Comedy movies
- 27 enjoyed watching all three
- 13 enjoyed watching none of these types of movies

Draw a Venn diagram to model this situation, and use it to determine how many total adults were surveyed, and how many of those surveyed enjoy Family movies.

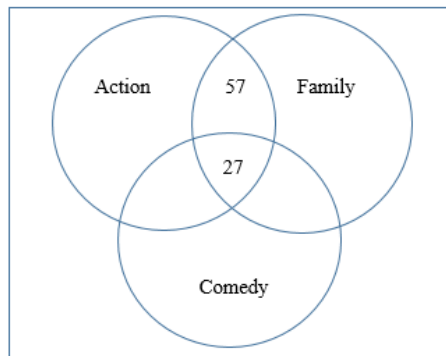
Solution: We have three sets to include in our Venn diagram: those who enjoy Action movies, those who enjoy Family movies, and those who enjoy Comedy movies. The initial set up for this Venn diagram looks like this:



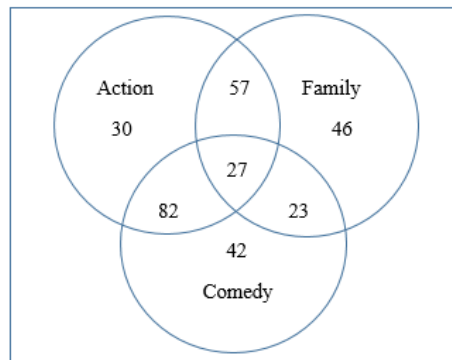
As in the previous example, we will start in the middle of the diagram, first labeling the cardinality for the intersection of all three sets. This represents those who enjoy Action, Family, and Comedy movies. According to the survey data, this was a total of 27 people. At this point, the Venn diagram looks as follows:



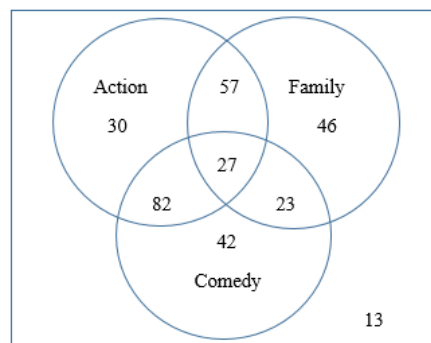
As is the case in Venn diagrams, once we have labeled in the center, we start working our way out of the middle. Consider where the Action and Family circles overlap. We are told that “57 enjoyed watching only Action and Family movies”. The wording “only” here is important to notice. When we use the word “only”, we are describing only that portion of the Venn diagram where Action and Family overlap (NOT INCLUDING COMEDY). The Venn diagram at this point becomes:



Continuing with the same idea, we can fill in the remaining segments in the Venn diagram. Again, it is important to note that this process works due to the word “only” being used to describe each portion of the data set.



Finally, we know 13 people enjoyed watching none of these types of movies. That means those 13 are in the universal set of those surveyed, but not inside any of the three sets. The finished Venn diagram looks as follows:



Answers: To answer the question of how many total adults were surveyed, we add up all the numbers from the Venn diagram.

$$30 + 57 + 46 + 82 + 27 + 23 + 42 + 13 = \mathbf{320 \text{ adults were surveyed}}$$

To answer the question of how many of those surveyed enjoyed Family movies, we add up all of the numbers inside the Family circle.

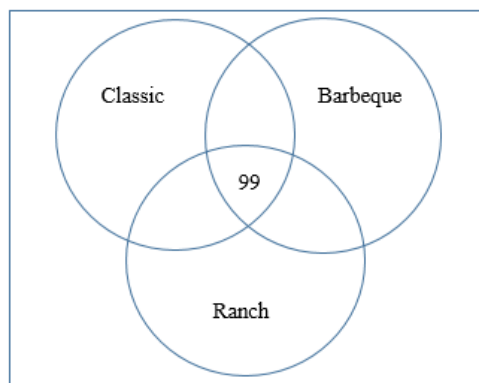
$$46 + 23 + 27 + 57 = \mathbf{153 \text{ adults enjoyed Family movies}}$$

Example 16: In a survey of 556 adults about consumer preferences for flavors of potato chips, the following results were obtained:

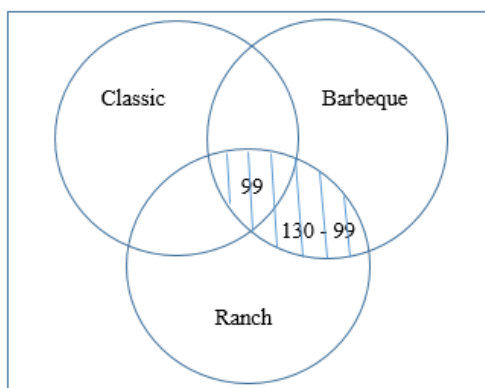
- 433 buy classic chips
- 277 buy barbeque chips
- 192 buy ranch chips
- 184 buy classic and barbeque chips
- 142 buy classic and ranch chips
- 130 buy barbeque and ranch chips
- 99 buy all three flavors

Use this information to draw a Venn diagram, and determine how many people don't buy any of these varieties of chips.

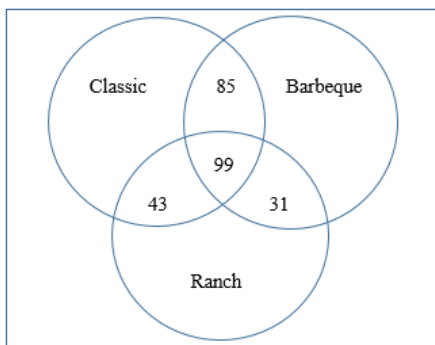
Solution: As with the previous example, we will start in the middle, with the intersection of all three sets. There are 99 adults surveyed who buy all three flavors, so 99 goes in the center of the Venn diagram:



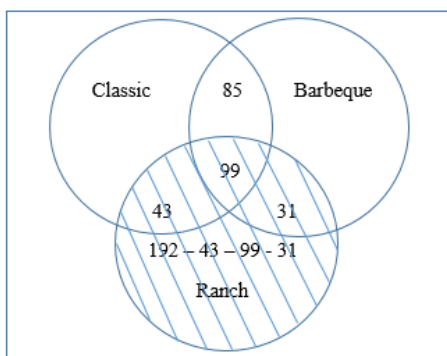
Once we have the number in the intersection of all three sets, we start working our way out from the center. We are told that 130 people buy barbeque and ranch chips. **An important distinction to make is that in this example, unlike the last, we are NOT told that 130 people ONLY buy barbeque and ranch.** The 130 people that buy barbeque and ranch is the shaded area as shown below, and includes the 99 already counted that buy all three.



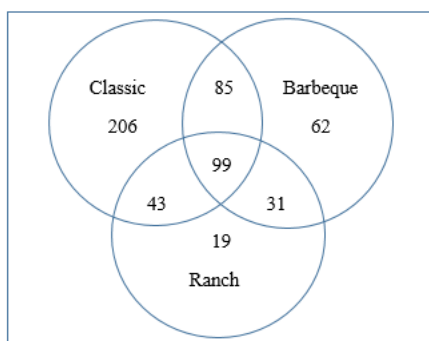
Use this same idea to fill in each of the regions of the Venn diagram where two of the sets overlap. The Venn diagram at this point is as follows:



Continue working your way outside, now taking the number of adults in each set individually. We are told that 192 buy Ranch chips. This 192 includes the entire Ranch circle as shaded below (of which, we already have 43, 99, and 31 of the adults accounted for from previous information).



Use this same idea to fill in the remaining segments of the Venn diagram.

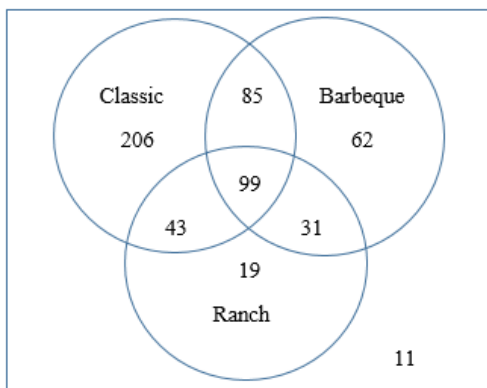


Finally, we know that 556 adults total were surveyed. This piece of information allows us to finish the Venn diagram, and answer the question “How many of those surveyed don’t buy any of these varieties of chips?”

Starting with the 556 surveyed, we subtract out all of the numbers from the known segments of the Venn diagram:

$$556 - 206 - 85 - 62 - 43 - 99 - 31 - 19 = 11 \text{ adults don't buy any of these varieties}$$

The final Venn diagram is as follows:



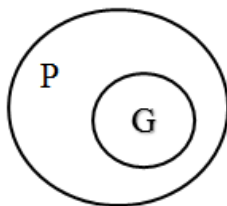
Chapter 7 – Answers to You Try Problems

7.1: There are several answers: The set of all odd numbers less than 10. The set of all odd numbers. The set of all integers. The set of all real numbers.

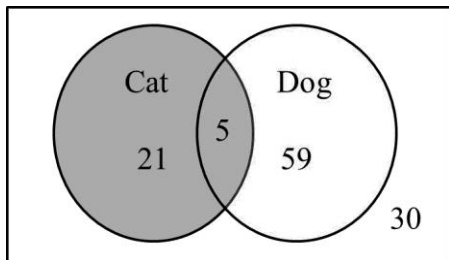
7.2A: $A \cup C = \{\text{red, orange, yellow, green, blue, purple}\}$ $B' \cap A = \{\text{green, blue}\}$

7.2B: $A' = \{5, 7, 13, 15\}$

7.3: G is a proper subset of P, so G must be completely contained within P.



7.4: 30 owned neither a cat nor a dog



CHAPTER 7 – PRACTICE PROBLEMS

- 1) List out the elements of the set “The letters of the word Mississippi”.
- 2) List out the elements of the set “Months of the year”.
- 3) Write a verbal description of the set $\{3, 6, 9\}$.
- 4) Write a verbal description of the set $\{a, i, e, o, u\}$.
- 5) Is $\{6, 8, 10\}$ a subset of the set of even integers?
- 6) Is $\{15, 23, 31\}$ a subset of the set of odd integers?
- 7) Is $\{A, B, C\}$ a subset of the set of letters of the alphabet?
- 8) Consider the sets:
 $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$ $D = \{\text{numbers from 0 to 10}\}$

Indicate if each statement is true or false.

- a) $3 \in B$
 - b) $5 \in C$
 - c) $B \subset A$
 - d) $C \subset A$
 - e) $C \subset B$
 - f) $A \subset D$
 - g) $A \subset B$
-
- 9) Consider the sets:
 $A = \{1, 3, 7\}$ $B = \{3, 5, 7, 9\}$ $C = \{1, 3, 4, 7, 8, 9\}$ $D = \{\text{odd from 0 to 10}\}$

Indicate if each statement is true or false.

- a) $10 \in D$
- b) $7 \in A$
- c) $A \subset C$
- d) $A \subset D$
- e) $B \subset C$
- f) $B \subset D$
- g) $C \subset D$

10) Let set A represent all students currently attending college in Arizona and let set B represent all students currently attending Scottsdale Community College. Use set notation to describe the relationship between the two sets A and B .

11) Let set A represent all cookies, let set B represent chocolate chip cookies, and let C represent all desserts. Indicate if each statement is true or false.

- a) "sugar cookies" $\in C$
- b) "sugar cookies" $\in B$
- c) "chocolate cake" $\in C$
- d) $B \subset A$
- e) $C \subset B$
- f) $A \subset C$

12) Suppose the universal set is $U = \{1, 2, 3, \dots, 8, 9, 10\}$. If $A = \{5, 6, 8\}$, find A' .

13) Suppose the universal set is $U = \{\text{red, orange, yellow, green, blue, purple}\}$. If $A = \{\text{red, green, blue}\}$, find A' .

14) Suppose the universal set U is all even numbers from 2 to 20.

If $A = \{4, 6, 10, 16, 18, 20\}$, find A'

15) Let $U = \{1, 2, 3, \dots, 18, 19, 20\}$ be the universal set. Consider the sets:

$$A = \{2, 3, 5, 7, 10, 11, 12, 13, 14, 16, 17, 19\} \quad B = \{5, 10, 13, 14, 16, 19, 20\}$$

Find each of the following:

- a) $A \cup B$
- b) $A \cap B$
- c) $A' \cap B$

16) Let $U = \{1, 2, 3, \dots, 18, 19, 20\}$ be the universal set. Consider the sets:

$$A = \{6, 11, 12, 14, 16, 17, 18\} \quad B = \{1, 4, 5, 8, 11, 12, 15, 16\}$$

Find each of the following:

- a) $A \cup B$
- b) $A \cap B$
- c) $A \cap B'$

17) Let $U = \{1, 2, 3, \dots, 18, 19, 20\}$ be the universal set. Consider the sets:

$$A = \{1, 2, 3, 4, 6, 8, 9, 11, 12, 14, 15, 18\} \quad B = \{5, 9, 12, 14, 17, 19, 20\}$$

Find each of the following:

- a) $A \cup B$
- b) $A \cap B$
- c) $A' \cap B$

18) Let $U = \{1, 2, 3, \dots, 8, 9, 10\}$ be the universal set. Consider the sets:

$$A = \{3, 5, 8, 10\} \quad B = \{1, 3, 4, 7, 8, 9, 10\} \quad C = \{2, 5, 6\}$$

Find each of the following:

- a) $A \cup C$
- b) $B \cap C$
- c) $A \cap B'$
- d) $(A \cup C)'$
- e) $(A \cap B)' \cup C$

19) Let $D = \{b, a, c, k\}$, $E = \{t, a, s, k\}$, $F = \{b, a, t, h\}$. Using these sets, find the following:

(Hint: first determine the elements in the universal set)

- a) $D' \cap E$
- b) $F' \cap D$
- c) $(D \cap E) \cup F$
- d) $D \cap (E \cup F)$
- e) $(F \cap E)' \cap D$
- f) $(D \cup E)' \cap F$

For **problems 20 – 25**, create a Venn diagram to illustrate each of the following.

- 20) C'
- 21) $A \cup B$
- 22) $A \cap B$
- 23) $A \cap B'$
- 24) $A \cup B \cup C$
- 25) $A \cap B \cap C$

26) Let $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 3, 5\}$ $C = \{4, 6\}$

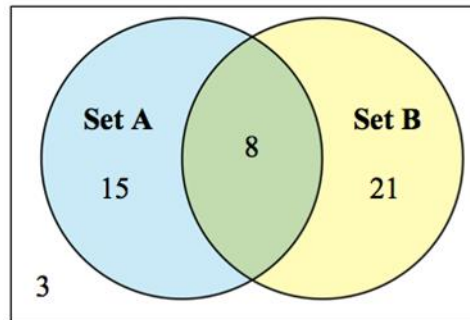
- a) What is the cardinality of A ?
- b) What is the cardinality of B ?
- c) What is the cardinality of $A \cup C$?
- d) What is the cardinality of $A \cap C$?

27) Let $A = \{1, 5, 9\}$ $B = \{1, 2, 3, 4, 5, 6, 8\}$ $C = \{2, 4, 5, 8, 9\}$

Determine the following:

- d) $n(A)$
- e) $n(C)$
- f) $n(B)$
- g) $n(B \cap C)$
- h) $n(A \cup C)$

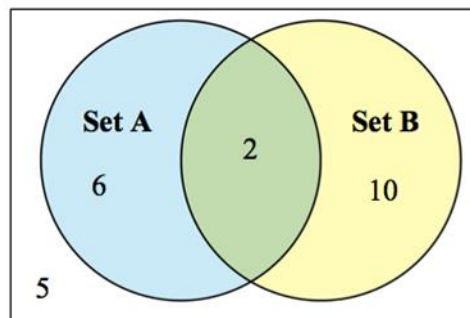
28) Consider the Venn diagram below.



Determine the following:

- a) $n(A)$
- b) $n(A')$
- c) $n(B)$
- d) $n(B')$
- e) $n(A \cap B)$
- f) $n(A \cup B)$
- g) $n(A' \cap B)$

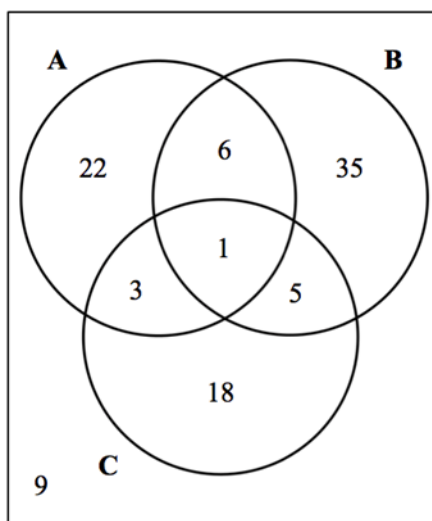
29) Consider the Venn diagram below.



Determine the following:

- a) $n(A)$
- b) $n(A')$
- c) $n(B)$
- d) $n(B')$
- e) $n(A \cap B)$
- f) $n(A \cup B)$
- g) $n(A' \cap B')$

30) Consider the Venn diagram below.

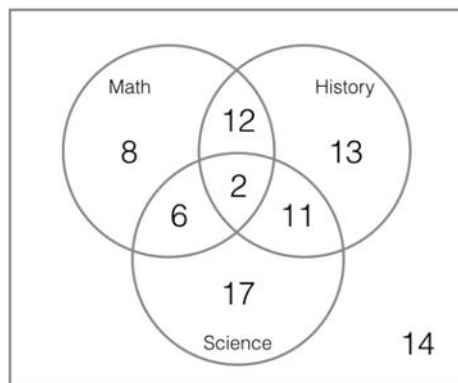


Determine the following:

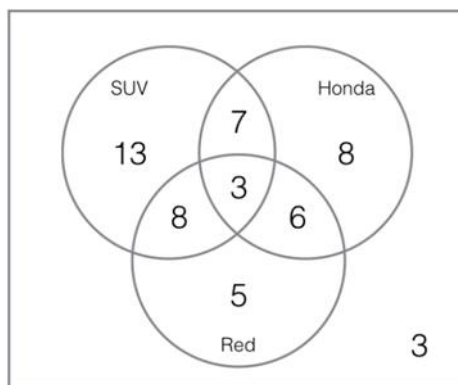
- $n(A)$
 - $n(C)$
 - $n(B')$
 - $n(A \cap C)$
 - $n(B \cup C)$
 - $n((B \cap C) \cup A)$
- 31) In a survey of 90 pet owners, 15 said they own a dog, and 49 said they own a cat. 6 said they own both a dog and a cat. How many owned a cat but not a dog?
- 32) In a survey of 121 pet owners, 55 said they own a dog, and 84 said they own a cat. 47 said they own both a dog and a cat. How many owned a dog but not a cat?
- 33) In a survey of 112 pet owners, 77 said they own a dog, and 33 said they own a cat. 20 said they own both a dog and a cat. How many owned neither a cat nor a dog?
- 34) Out of 60 students, 21 are taking English, 34 are taking Math, and 7 are in both classes.
- Draw a Venn diagram to organize the information.
 - How many students are in either English or Math?
 - How many students are not in English?
 - How many students are in English and not in Math?
- 35) Students who take AP English also take the AP test, of which they either pass or fail. Of the 28 students taking AP English, 10 received an A in the course. Out of the 15 students that passed the AP exam, 9 received an A.
- Draw a Venn diagram to organize the information.
 - How many students failed the AP exam?
 - How many students passed the AP exam and did not receive an A?
 - How many students failed the AP exam and did not receive an A?

- 36) A team has 39 baseball games this season. Out of the 22 home games, the team won 14 of them. The team had a total of 25 wins over the entire season.
- Draw a Venn diagram to organize the information.
 - How many of the away games did the team win?
 - How many of the away games did the team lose?

37) The following Venn diagram shows students in various college courses.

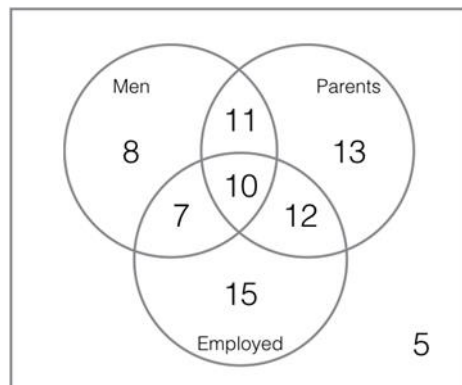


- How many students are taking Math?
 - How many students are taking Math and Science?
 - How many students are taking History and Science, but not Math?
- 38) The following Venn diagram shows cars in a parking lot.



- How many Hondas are in the parking lot?
- How many red SUVs are in the parking lot?
- How many of the Red cars in the parking lot are not Hondas?

39) The following Venn diagram shows people at a party.



- How many people at the party are employed?
- How many people at the party are mothers?
- How many people at the party are working fathers?
- How many people at the party are not parents?

40) A survey was given to students asking whether they watch movies at home from Netflix, Redbox, or a video store. Use the results to determine how many people use Redbox.

52 only use Netflix	62 only use Redbox
24 only use a video store	16 use only a video store and Redbox
48 use only Netflix and Redbox	30 use only a video store and Netflix
10 use all three	25 use none of these

41) A survey asked buyers whether color, size, or brand influenced their choice of cell phone. The results are below. How many people were influenced by brand?

5 only said color	8 only said size
16 only said brand	20 said only color and size
42 said only color and brand	53 said only size and brand
102 said all three	20 said none of these

42) A survey asked people what alternative transportation modes they use. Use the data to complete a Venn diagram, then determine:

- what percent of people only ride the bus
 - how many people don't use any alternate transportation
- | | |
|-----------------------------|-----------------------------------|
| 30% use the bus | 20% ride a bicycle |
| 25% walk | 5% use the bus and ride a bicycle |
| 10% ride a bicycle and walk | 12% use the bus and walk |
| 2% use all three | |

CHAPTER 8 – INTRODUCTION TO PROBABILITY

Chapter 8 Learning Objectives:

- 1) Distinguish between subjective, experimental, and theoretical probabilities
- 2) Be familiar with probability terminology and properties of probabilities
- 3) Use the complement formula to find probabilities
- 4) Create tables to calculate expected values for real world applications including insurance and product warranties

Sections

Section 8.1 – Types of Probability

Section 8.2 – Basic Concepts

Section 8.3 – Complementary Events

Section 8.4 – Expected Value

SECTION 8.1: TYPES OF PROBABILITY

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much. Let's consider two other types of probabilities.

Another view of probability would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

We will return to the subjective and experimental probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows:

Theoretical Probability

Suppose there is a situation with n **equally likely** possible outcomes and that m of those n outcomes correspond to a particular event; then the probability of that event is defined as $\frac{m}{n}$.

$$\text{Probability of an event} = \frac{\text{number of ways the event can occur}}{\text{total number of outcomes}}$$

Let's return to the coin flipping example. Instead of calculating the probability of the coin landing on heads by flipping a coin many times and observing the number of heads (which is an experimental probability) we could instead use what we know about coins. A fair coin has 2 equally likely possible outcomes: heads or tails. If we are interested in calculating the theoretical probability of a coin landing on heads, we could do so using the formula above:

$$\frac{\text{number of ways a head can occur}}{\text{number of possible outcomes}} = \frac{1}{2}$$

SECTION 8.2: BASIC CONCEPTS

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further.

The **sample space** is the set of all possible outcomes.

Example 1: If we roll a standard 6-sided die, describe the sample space and some simple events.

Solution: The sample space is the set of all possible outcomes: $\{1, 2, 3, 4, 5, 6\}$

Some examples of simple events:

We roll a 1. We roll a 5.

An example of an event:

We roll a number bigger than 4.



Two dice



One die

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{number of outcomes corresponding to the event } E}{\text{total number of equally likely outcomes}}$$

Example 2: If we roll a 6-sided die, calculate

a) $P(\text{rolling a 1})$

b) $P(\text{rolling a number bigger than 4})$

Solution: Recall that the sample space is $\{1, 2, 3, 4, 5, 6\}$

a) There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$.

b) There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$.

Section 8.2A You Try: If we roll a 6-sided die, calculate $P(\text{rolling an odd number})$.

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3: Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution: There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Section 8.2B You Try: At some random moment, you look at your clock and note the minutes reading.

- What is probability the minutes reading is 15?
- What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King. The Jack, Queen, and King are called face cards.

K	Q	J	10	9	8	7	6	5	4	3	2	A	Clubs black
♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	♣	
K	Q	J	10	9	8	7	6	5	4	3	2	A	Spades black
♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	♠	
K	Q	J	10	9	8	7	6	5	4	3	2	A	Hearts red
♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	♥	
K	Q	J	10	9	8	7	6	5	4	3	2	A	Diamonds red
♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	♦	

suits: clubs, spades, hearts, diamonds (13 per suit)
 face cards: King (K), Queen (Q), Jack (J) (3 per suit)
 total cards: 52

Example 4: Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution: There are 52 cards in the deck and 4 Aces so $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$. We can also think of probabilities as percentages: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 (there are no outcomes that correspond with the event). The largest possible probability is 1 (all possible outcomes correspond with the event).

Certain and Impossible events

An **impossible** event has a probability of 0. A **certain** event has a probability of 1. The probability of any event must be between or equal to 0 or 1, or $0 \leq P(E) \leq 1$.

In the course of this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Example 5: Find the probability of drawing a single card from a deck and getting a red spade.

Solution: There are 52 cards in the deck, but all spades are black. It is impossible to draw a single card and have that card be a red spade. This is an impossible event, so

$$P(\text{red spade}) = \frac{0}{52} = 0.$$

Example 6: You wake up on a cloudy, drizzly Seattle day to the weatherman saying the chance of rain today is 100%. Find the probability that it will rain today.

Solution: Since we are told the chance of rain is 100%, we know that this is a certain event.

$$P(\text{rain}) = \frac{100\%}{100\%} = 1.$$

Section 8.2C You Try: Your neighbor tells you that he heard on the radio that the probability of an earthquake of magnitude 7.0 or higher hitting California in the next 10 years is 1.2. How do you know that this is incorrect?

SECTION 8.3: COMPLEMENTARY EVENTS

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = 1/6$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = 5/6$.

Notice that:

$$P(6) + P(\text{not } 6) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence.

Complement of an Event

The complement of an event is the event “ E does **not** happen”. The notation E' is used for the complement of event E . You should recognize this notation from the last chapter.

For any event E ,

$$P(E) + P(E') = 1$$

We can compute the probability of the complement using $P(E') = 1 - P(E)$. Notice also that $P(E) = 1 - P(E')$

Example 7: If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution: There are 13 hearts in the deck, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$

The probability of **not** drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Example 8: A jar contains 28 marbles, 12 of which are red. If you pick 1 marble out of the jar, what is the probability that it is not red?

Solution: There are 12 red marbles out of 28 total marbles, so $P(\text{red}) = \frac{12}{28} = \frac{3}{7}$

The probability of *not* drawing a red marble is the complement:

$$P(\text{not red}) = 1 - P(\text{red}) = 1 - \frac{3}{7} = \frac{4}{7}.$$

Section 8.3 You Try: Your favorite basketball player is an 84% free throw shooter. Find the probability that he does NOT make his next free throw.

SECTION 8.4: EXPECTED VALUE

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it is one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 9: ⁶In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?



Solution: Suppose you bet \$1 on each of the 38 spaces on the wheel, for a total of \$38 bet. When the winning number is spun, you are paid \$36 on that number. While you won on that one number, overall you've lost \$2. On a per-space basis, you have "won" $-\$2/\$38 \approx -\$0.053$. In other words, on average, you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose \$1 and a very few people (about 1 person out of every 38) gain \$35 (the \$36 they win minus the \$1 they spent to play the game).

Example 10: Refer back to the roulette example. The wheel that is spun has 38 spaces (18 red, 18 black, and 2 green). In one possible bet, the player bets \$1 on a single number. If that number is spun on the wheel, then they receive \$36 (their original \$1 + \$35). Otherwise, they lose their \$1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Solution: There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The probability of losing, is $\frac{37}{38}$.

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \times \frac{1}{38} = \0.9211 and $-\$1 \times \frac{37}{38} = -\0.9737 , and if we add these numbers we get

$\$0.9211 + (-\$0.9737) \approx -\$0.053$, which is the expected value of the game.

⁶ Photo CC-BY-SA <http://www.flickr.com/photos/stoneflower/>

We can place this process in table form as shown below. We find the sum of all the products of Outcome $\times$ Probability to determine the expected value.

Outcome	Probability of Outcome	Outcome $\times$ Probability
\$35	$\frac{1}{38}$	$\$35 \times \frac{1}{38} = \0.9211
-\$1	$\frac{37}{38}$	$-\$1 \times \frac{37}{38} = -\0.9737
		Expected Value: $\$0.9211 + (-\$0.9737) \approx -\$0.053$

Answer: If we play roulette many times, we can expect that on average we will lose \$0.053 per play. This expected value is negative, which indicates that on average we will lose money.

Definition: Expected Value is the average gain or loss of an event if the procedure is repeated many times. We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the *average* winnings are positive it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a **fair game**, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player, which is positive for the casino. It must be positive or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Section 8.4 You Try: A raffle ticket costs \$3. There is 1 winning ticket out of the 150 tickets sold. The winner gets \$300. What is the expected value (to you) of the raffle ticket? Should you buy a ticket?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 11: A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year⁷. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

Solution: The probabilities and outcomes are

Outcome	Probability of Outcome	Outcome $\times$ Probability
$\$1000 - \$275 = \$99,725$	0.00242	\$241.33
$-\$275$	$1 - 0.00242 = 0.99758$	$-\$274.33$
		Expected Value = $-\$33$

Answer: The expected value for the person buying the insurance is $-\$33$

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people. For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

Example 12: Consider the same problem we just did, but we will answer it using a slightly different setup.

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year⁸. An insurance company charges \$275 for a life-insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance?

Solution: The man will pay \$275 for the policy whether or not he dies in the next year, so the probability of losing the \$275 is 100%.

Outcome	Probability of Outcome	Outcome $\times$ Probability
$-\$275$	1	$-\$275$
\$100,000	0.00242	\$242
		Expected Value = $-\$33$

Answer: The expected value is $(-\$275)(1) + (\$100,000)(0.00242) = -\$33$

⁷ According to the estimator at <http://www.numericaexample.com/index.php?view=article&id=91>

⁸ According to the estimator at <http://www.numericaexample.com/index.php?view=article&id=91>

Example 13: A company estimates that 0.3% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$500. If they offer a 2 year extended warranty for \$40, what is the company's expected value of each warranty sold?

Solution: The company will receive the \$40 for the warranty whether or not the product fails, so the probability of receiving the \$40 is 100%. The company has a 0.3% chance of having to pay the \$500 replacement cost if the product fails.

The expected value (to the company) is $\$40(1) + (-\$500)(0.003) = \$38.50$

Answer: On average, the company will earn \$38.50 for each policy sold.

Chapter 8 – Answers to You Try Problems

8.2A: $P(\text{rolling an odd}) = \frac{3}{6} = \frac{1}{2}$

8.2B: a) $\frac{1}{60}$ b) $\frac{16}{60} = \frac{4}{15}$

8.2C: Probabilities must be between 0 and 1, so a probability of 1.2 is impossible.

8.3: $1 - 0.84 = 0.16$

8.4: The expected value of the raffle ticket is $-\$1$. You should expect to lose \$1 for each ticket that you buy. You shouldn't buy a ticket.

CHAPTER 8 – PRACTICE PROBLEMS

- 1) Suppose you roll a six-sided die. Determine the following:
 - a) $P(\text{rolling a } 5)$
 - b) $P(\text{rolling an even number})$
 - c) $P(\text{rolling a number bigger than } 2)$
- 2) Compute the probability of tossing a six sided die and getting a 7.
- 3) Compute the probability of tossing a six sided die and getting a number less than 3.
- 4) Compute the probability of rolling a 12-sided die and getting a number other than 8.
- 5) A test was given to a group of students. The grades and gender are summarized below.

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

- a) Find the probability that a student chosen at random was female.
 - b) Find the probability that a student chosen at random did NOT earn a C?
- 6) The table below shows the number of credit cards owned by a group of individuals.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

- a) Find the probability that a person chosen at random had no credit cards.
 - b) Find the probability that a person chosen at random has at least one credit card?
- 7) A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
 - a) A red ball is drawn
 - b) A white ball is drawn
- 8) A bag contains 2 gold marbles, 10 silver marbles, and 25 black marbles. You randomly select one marble from the bag. What is the probability that you select a gold marble?

College Math – Chapter 8 Practice Problems

- 9) Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
- a) A consonant
 - b) A vowel
- 10) A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.
- 11) In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dogs, and 40 indicated they don't enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
- 12) If you pick one card at random from a standard deck of 52 cards, what is the probability it will be a King?
- 13) If you pick one card at random from a standard deck of 52 cards, what is the probability it will be a Diamond?
- 14) If you pick one card at random from a standard deck of 52 cards, what is the probability it is not the Ace of Spades?
- 15) If you pick one card at random from a standard deck of 52 cards, what is the probability it is not a heart?
- 16) If you pick one card at random from a standard deck of 52 cards, what is the probability it will be an 8?
- 17) If you pick one card at random from a standard deck of 52 cards, what is the probability it is not a 6?
- 18) If you pick one card at random from a standard deck of 52 cards, what is the probability it will be a face card?
- 19) If the probability of a given event is $\frac{24}{61}$, what is the probability of the event not happening?
- 20) If the probability of a given event not happening is $\frac{38}{45}$, what is the probability of the event happening?

- 21) A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select one marble from the bag. If it is gold, you win \$3. If it is silver, you win \$2. If it is black, you lose \$1. What is your expected value if you play this game?
- 22) A bag contains 2 gold marbles, 10 silver marbles, and 25 black marbles. Someone offers to play this game: You randomly select one marble from the bag. If it is gold, you win \$3. If it is silver, you win \$2. If it is black, you lose \$1. What is your expected value if you play this game?
- 23) A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you \$3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him \$1. Compute the expected value for this game. Should you play this game?
- 24) A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$350. If they offer a 2 year extended warranty for \$48, what is the company's expected value of each warranty sold?
- 25) An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be \$60,000. If the company offers earthquake insurance for \$100, what is their expected value of the policy?
- 26) A company estimates that 0.9% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of \$500. If they offer a 2 year extended warranty for \$45, what is the company's expected value of each warranty sold?

CHAPTER 9 – PROBABILITY

Chapter 9 Learning Objectives:

- 1) Be able to calculate “and”/“or” probabilities
- 2) Determine if two or more events are independent
- 3) Find conditional probabilities
- 4) Determine if two or more events are mutually exclusive

Sections

Section 9.1 – Independent Events

Section 9.2 – Conditional Probability

Section 9.3 – “Or” Probabilities

SECTION 9.1: INDEPENDENT EVENTS

Example 1: Suppose we flipped a coin and rolled a die, and wanted to know the probability of the coin landing on heads and rolling a 6 on the die.

Solution: We could list all possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}.

Notice there are $2 \times 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $P(\text{head and } 6) = \frac{1}{12}$.

The prior example was looking at two **independent events**. The events “landing on heads” and “rolling a 6” are independent because the outcome of the die roll does not affect the outcome of the coin flip.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 2: Are these events independent?

- A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
- You draw a card from a deck, then draw a second card without replacing the first. The two events are (1) the first card is red and (2) the second card is also red.

Solution:

a) The probability that a head comes up on the second toss is $1/2$ regardless of whether or not a head came up on the first toss, so these events are independent.

b) The probability of the first card being red is $P(\text{first card is red}) = \frac{26}{52} = \frac{1}{2}$.

Since the first card is not returned to the deck, the probability of the second card being red is not the same as the probability that the first card is red. After drawing the first card, there are only 51 cards remaining in the deck. If the first card was red, then there are only 25 red cards remaining, and the probability that the second card would be red is

$$P(\text{second card is red}) = \frac{25}{51}.$$

If the first card was not red, then the probability that the second card would be red is

$$P(\text{second card is red}) = \frac{26}{51}. \text{ These events are **not** independent.}$$

When two events are independent, the probability of both occurring is just the product of the probabilities of the individual events.

P(A and B) for independent events

If events A and B are independent, then the probability of both A and B occurring is

$$P(A \text{ and } B) = P(A) \times P(B)$$

where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring.

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 3: In Alex's drawer she has 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If she randomly reaches in and pulls out a pair of socks and a tee shirt, what is the probability that both are white?

Solution: The probability of choosing a white pair of socks is $\frac{6}{10} = \frac{3}{5}$.

The probability of choosing a white tee shirt is $\frac{3}{7}$.

The probability of both being white is $\frac{3}{5} \times \frac{3}{7} = \frac{9}{35}$.

Section 9.1 You Try: A couple has two children. Find the probability that both children are girls.

Example 4: You roll two die. Find the probability that the first die lands on an even number, and the second die lands on 5.

Solution: Since the outcome of the first die has no effect on the outcome of the second die, the events are independent.

$$P(\text{first die is even}) = \frac{3}{6} = \frac{1}{2}$$

$$P(\text{second die lands on 5}) = \frac{1}{6}$$

$$P(\text{first die is even AND second die lands on 5}) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

As a check, it may be helpful to consider all possible outcomes in the experiment of rolling 2 die:

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Notice there are 3 outcomes (out of 36 possible outcomes) where the first die is even and the second die lands on 5. We could have found this probability directly.

SECTION 9.2: CONDITIONAL PROBABILITY

Often it is required to compute the probability of an event given that another event has occurred.

Example 5: What is the probability that two cards drawn at random from a deck of playing cards will both be aces?

Solution: It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are **not independent**. Once the first card is drawn, there are only 51 cards remaining in the deck. If the first card drawn is an ace, then the probability that the second card is also an ace would be because there would only be three aces left in the deck.

Once the first card chosen is an ace, the probability that the second card chosen is also an ace is called the **conditional probability** of drawing an ace. In this case, the "condition" is that the first card is an ace. Symbolically, we write this as:

$$P(\text{ace on second draw} \mid \text{an ace on the first draw}).$$

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an ace is drawn on the second draw given that an ace was drawn on the first draw." What is this probability? After an ace is drawn on the first draw, there are 3 aces out of 51 total cards left. This means that the conditional probability of drawing an ace after one ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being aces is $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$.

Conditional Probability

The probability that event B occurs, given that event A has happened, is represented as $P(B \mid A)$. This is read as "the probability of B given A "

Example 6: Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution: These are two independent events, so the probability of the die rolling a 6 is $\frac{1}{6}$, regardless of the result of the coin flip.

Example 7: The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their cars. Find the probability that a randomly chosen person:

- Has a speeding ticket *given* they have a red car
- Has a red car *given* they have a speeding ticket

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution:

a) Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ticket} \mid \text{red car}) = \frac{15}{150} = \frac{1}{10} = 0.1$$

b) Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{red car} \mid \text{ticket}) = \frac{15}{60} = \frac{1}{4} = 0.25$$

Notice from the last example that $P(B \mid A)$ is not equal to $P(A \mid B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having an accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

$P(A \text{ and } B)$ for dependent events

If Events A and B are dependent (not independent), then $P(A \text{ and } B) = P(A) \times P(B \mid A)$

Example 8: If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution: The probability that the first card is a spade is $\frac{13}{52}$. The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards.

The probability that both cards are spades is $\frac{13}{52} \times \frac{12}{51} = \frac{156}{2652} \approx 0.0588$

Section 9.2 You Try: If you draw 2 cards from a standard deck, what is the probability that both cards are red kings?

Example 9: A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

- a) $P(\text{not pregnant} \mid \text{positive test result})$
- b) $P(\text{positive test result} \mid \text{not pregnant})$

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

Solution:

a) Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.

$$P(\text{not pregnant} \mid \text{positive test result}) = \frac{5}{75} \approx 0.067.$$

b) Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

$$P(\text{positive test result} \mid \text{not pregnant}) = \frac{5}{19} \approx 0.263$$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant.

SECTION 9.3: "OR" PROBABILITIES

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 10: Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution: Here, there are still 12 possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability.

Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

P(A or B)

The probability of either A or B occurring (or both) is: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example 11: Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution: There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{King or Queen}) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so:

$$P(\text{King and Queen}) = 0$$

Using our probability rule, we could have said:

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 12: Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution: Half the cards are red, so $P(\text{red}) = \frac{26}{52}$

There are four kings, so $P(\text{King}) = \frac{4}{52}$

There are two red kings, so $P(\text{red and King}) = \frac{2}{52}$

We can then calculate

$$P(\text{red or King}) = P(\text{red}) + P(\text{King}) - P(\text{red and King}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

Example 13: Consider again that we roll two die. What is the probability that the **sum** of the two die is odd or 4?

Solution: Below is a list of all possible outcomes in the experiment of rolling 2 die:

1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Note: The table gives the values of each die, not the sum of the two die.

Since it is not possible for the sum to be *both* odd and 4, these events are mutually exclusive.

$$P(\text{sum is odd}) = \frac{18}{36} = \frac{1}{2}$$

$$P(\text{sum is 4}) = \frac{3}{36}$$

$$P(\text{sum is odd and 4}) = \frac{0}{36}$$

$$P(\text{sum is odd or 4}) = P(\text{sum is odd}) + P(\text{sum is 4}) - P(\text{sum is odd and 4}) =$$

$$\frac{18}{36} + \frac{3}{36} - \frac{0}{36} = \frac{21}{36} = \frac{7}{12}$$

Section 9.3 You Try: Consider again that we roll two die. What is the probability that the sum of the two die is odd or one of the die is a 1?

Example 14: The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

- Has a red car *and* got a speeding ticket
- Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Solution:

a) We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is

$$P(\text{red car and speeding ticket}) = \frac{15}{665} \approx 0.0226$$

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

b) We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is

$$P(\text{red car or speeding ticket}) = \frac{195}{665} \approx 0.2932$$

We also could have found this probability by:

$$P(\text{red car}) + P(\text{speeding ticket}) - P(\text{red car and speeding ticket}) =$$

$$\frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665} \approx 0.2932$$

Chapter 9 – Answers to You Try Problems

9.1: $1/4$

9.2: 0.00075

9.3: $23/36$

CHAPTER 9 – PRACTICE PROBLEMS

- 1) A six sided die is rolled twice. What is the probability of showing a 2 on the first roll and an odd number on the second roll?
- 2) A six sided die is rolled three times. What is the probability of showing an even number on all three rolls?
- 3) A six sided die is rolled twice. What is the probability of showing a 6 on both rolls?
- 4) A fair coin is flipped twice. What is the probability of showing heads on both flips?
- 5) A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?
- 6) A couple has three children. What is the probability that all three are girls?
- 7) Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?
- 8) In your drawer you have 5 pairs of socks, 4 of which are white, and 11 tee shirts, 2 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability that both are white?
- 9) Bert and Ernie each have a well shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
 - a) Bert and Ernie both draw an Ace.
 - b) Bert draws an Ace but Ernie does not.
 - c) Neither Bert nor Ernie draws an Ace.
 - d) Bert and Ernie both draw a heart.
 - e) Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.
- 10) Bert has a well shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12–sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
 - a) Bert gets a Jack and Ernie rolls a five.
 - b) Bert gets a heart and Ernie rolls a number less than six.
 - c) Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
 - d) Bert gets a red card and Ernie rolls a fifteen.
 - e) Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

College Math - Chapter 9 Practice Problems

- 11) A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
- a) Even numbered given that the marble is red.
 - b) Red given that the marble is even numbered.
- 12) A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
- a) Odd numbered given that the marble is blue.
 - b) Blue given that the marble is odd numbered.
- 13) Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.
- 14) Find the probability of rolling a "1" on a fair die, given that the last 3 rolls were all ones.
- 15) Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French).
- a) Compute the probability that a randomly selected student speaks French, given that the student is female.
 - b) Compute the probability that a randomly selected student is male, given that the student speaks French.
- 16) A test was given to a group of students. The grades and gender are summarized below.

	A	B	C
Male	10	12	2
Female	16	6	9
Total	26	18	11

Suppose a student is chosen at random:

- a) Find the probability that the student was male given they earned an A.
- b) Find the probability that the student was male given they earned a C.
- c) Find the probability that the student was female given they earned a B.

17) A test was given to a group of students. The grades and gender are summarized below.

	A	B	C
Male	7	6	4
Female	9	2	10
Total	16	8	14

Suppose a student is chosen at random:

- Find the probability that the student earned a B given they are male.
- Find the probability that the student earned a B given they are female.
- Find the probability that the student earned a C given they are male.

18) A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

- Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A | B)$.
- Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{not } A | \text{not } B)$.

19) A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".

- Find the probability that a person has the virus given that they have tested positive, i.e. find $P(A | B)$.
- Find the probability that a person does not have the virus given that they test negative, i.e. find $P(\text{not } A | \text{not } B)$.

20) Two cards are drawn from a standard deck of cards. What is the probability of drawing a King and then drawing a Queen.

21) Two cards are drawn from a standard deck of cards. What is the probability of both cards being red?

22) Two cards are drawn from a standard deck of cards. What is the probability of drawing a Jack and then drawing an Ace?

- 23) Five cards are drawn from a standard deck of cards. What is the probability of all cards being black?
- 24) Tony buys a bag of cookies that contains 4 chocolate chip cookies, 9 peanut butter cookies, 7 sugar cookies and 8 oatmeal cookies. What is the probability that Tony reaches in the bag and randomly selects a peanut butter cookie from the bag, eats it, then reaches back in the bag and randomly selects a sugar cookie?
- 25) Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.
- 26) Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.
- 27) A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that
- a male is selected, then a female.
 - a female is selected, then a male.
 - two males are selected.
 - two females are selected.
 - no males are selected
- 28) A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that
- a male is selected, then two females.
 - a female is selected, then two males.
 - two females are selected, then one male.
 - three males are selected.
 - three females are selected.
- 29) A test was given to a group of students. The grades and gender are summarized below.

	A	B	C
Male	8	18	13
Female	10	4	12
Total	18	22	25

Suppose a student is chosen at random:

- Find the probability that the student is female and earned an A.
- Find the probability that the student is male and earned an A.
- Find the probability that the student is male or earned a C.
- Find the probability that the student is female or earned a B.

30) The table below shows the number of credit cards owned by a group of individuals.

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

Suppose a person is chosen at random:

- Find the probability that the person is male and has two or more credit cards.
- Find the probability that the person is female and has one credit card.
- Find the probability that the person is male or has no credit cards.
- Find the probability that the person is female or has two or more credit cards.

31) Suppose we draw one card from a standard deck. What is the probability of drawing the King of hearts or a Queen?

32) Suppose you roll a six sided die. What is the probability that you roll an even number or a five?

33) Suppose we draw one card from a standard deck. What is the probability of drawing a King or a heart?

34) Suppose we draw one card from a standard deck. What is the probability of drawing a face card or a diamond?

35) Suppose we draw one card from a standard deck. What is the probability of drawing a Jack or a black card?

36) Two six sided dice are rolled. What is the probability of getting a sum of either 11 or 12?

37) A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd numbered.

38) A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even numbered.

39) A jar contains 4 red marbles numbered 1 to 4 and 12 blue marbles numbered 1 to 12. A marble is drawn at random from the jar. Find the probability of the given event.

- The marble is red or odd numbered
- The marble is blue and even numbered

40) Given the following information, determine $P(A \text{ or } B)$

$$P(A) = 0.56$$

$$P(B) = 0.53$$

$$P(A \text{ and } B) = 0.42$$

$$P(B \mid A) = 0.3$$

41) Given the following information, determine $P(A \text{ or } B)$

$$P(A) = 0.76$$

$$P(B) = 0.3$$

$$P(A \text{ and } B) = 0.28$$

$$P(B \mid A) = 0.2$$

CHAPTER 10 – COUNTING

Chapter 10 Learning Objectives:

- 1) Use tables and tree diagrams to model possible outcomes in counting problems
- 2) Use permutations and combinations for counting
- 3) Be able to find probabilities using permutations and combinations

Section 10.1 – Basic Counting

Section 10.2 – Permutations

Section 10.3 – Combinations

Section 10.4 – Finding Probabilities Using Counting

SECTION 10.1: BASIC COUNTING

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting *efficiently*. When we get to the probability situations a bit later in this chapter we will need to count some *very* large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 1: Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger
soup + fajita
salad + sandwich
salad + pizza
breadsticks + quiche

soup + sandwich
soup + pizza
salad + quiche
breadsticks + hamburger
breadsticks + fajita

soup + quiche
salad + hamburger
salad + fajita
breadsticks + sandwich
breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

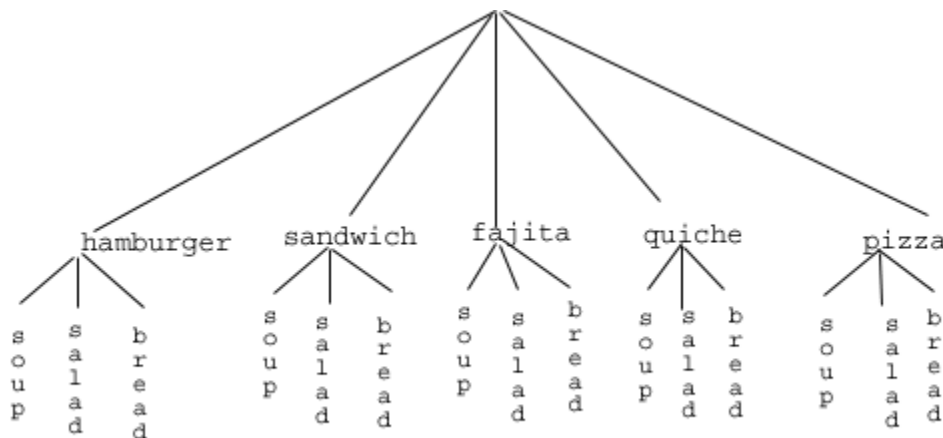
Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

	hamburger	sandwich	quiche	fajita	pizza
soup	soup+burger				
salad	salad+burger				
bread	etc.				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care *what* the possible meals are, only *how many* possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:



This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the *basic counting rule*:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \times n$. This is sometimes called the multiplication rule for probabilities.

Example 2: There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution: There are 21 choices from the first category and 18 for the second, so there are $21 \times 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 3: Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution: There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are $3 \times 5 \times 2 = 30$ possibilities.

Section 10.1 You Try: You are making your schedule for next semester. You have 5 choices for your math class, 8 choices for your English class, 3 choices for your communications class, and 4 choices for your elective. How many different schedules are possible?

Example 4: A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution: There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \times 2 \times 2 = 8$ different ways. Recall that another way to write $2 \times 2 \times 2$ is 2^3 , which is much more compact.

SECTION 10.2: PERMUTATIONS

We will now develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 5: How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution: This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \times 3 \times 2 \times 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \times (n - 1) \times (n - 2) \cdots 3 \times 2 \times 1$. This calculation shows up often in mathematics, and is called the **factorial**, and is notated $n!$

Example 6: How many ways can five different door prizes be distributed among five people?

Solution: There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Section 10.2A You Try: A little league baseball team has 12 players. How many different batting orders of the 12 players are possible?

Now we will consider some slightly different examples.

Example 7: A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution: Using the Basic Counting Rule, there are 25 choices for the person who receives the \$100 certificate, 24 remaining choices for the \$25 certificate and 23 choices for the \$10 certificate, so there are $25 \times 24 \times 23 = 13,800$ ways in which the prizes can be awarded.

Example 8: Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution: Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are $8 \times 7 \times 6 = 336$ ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order there might have been a different person who received the \$100 prize, or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a \$10 gift certificate; in this case the order of selection is *not* important since each of the three people receive the same prize. Situations where the order is *not* important will be discussed in the next section.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by

$$n \times (n - 1) \times (n - 2) \cdots (n - r + 1)$$

If you don't see why $(n - r + 1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated $25 \cdot 24 \cdot 23$ to get 13,800. In this case $n = 25$ and $r = 3$, so $n - r + 1 = 25 - 3 + 1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often, we only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutation Formula:

$${}_nP_r = n \times (n - 1) \times (n - 2) \cdots (n - r + 1)$$

We say that there are ${}_nP_r$ **permutations** of size r that may be selected from among n choices **without replacement** when **order matters**.

It turns out that we can express this result more simply using factorials.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 9: I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

Solution: Since we are choosing 4 paintings out of 9 *without replacement* where the *order of selection is important* there are ${}_9P_4 = 9 \times 8 \times 7 \times 6 = 3,024$ permutations.

Calculator: To enter ${}_9P_4$ in your graphing calculator, type 9, then hit the MATH key, scroll over to PRB, select option “2:nPr” by hitting enter. Your main screen should now show “9 nPr” and have a blinking cursor. Type 4, so that your calculator shows “9 nPr 4”. Hit enter and you will get the answer 3,024.

Example 10: How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution: We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \times 15 \times 14 \times 13 = 43,680$.

Section 10.2B You Try: How many 5 character passwords can be made using the letters A through Z

- a) if repeats are allowed
- b) if no repeats are allowed

SECTION 10.3: COMBINATIONS

In the previous section we considered the situation where we chose r items out of n possibilities *without replacement* and where the *order of selection was important*. We now consider a similar situation in which the order of selection is *not* important.

Example 11: A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution: Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are $25 \times 24 \times 23 = 13,800$ ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get \$50, so it doesn't really matter the order in which we select them. In how many different orders can Abe, Bea and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are $3 \times 2 \times 1 = 6$ ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea and David or Frank, Gloria and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be $13,800/6 = 2300$.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is not important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by, $\frac{{}_nP_r}{{}_rP_r}$ and we could use this formula for computation.

However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection is not important*.

Combinations Formula: ${}_nC_r = \frac{{}_nP_r}{{}_rP_r}$

We say that there are ${}_nC_r$ combinations of size r that may be selected from among n choices ***without replacement*** where ***order doesn't matter***.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

Example 12: A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution: Since we are choosing 4 people out of 35 without replacement where the order of

selection is ***not*** important there are ${}_{35}C_4 = \frac{35 \times 34 \times 33 \times 32}{4!} = 52,360$ combinations.

Calculator: Your graphing calculator will also find combinations. For this example, type 35, select MATH, PRB, nCr. Your main screen should show "35 nCr" with a blinking cursor. Type 4. Your main screen should now show "35 nCr 4". Hit "enter" to get the answer 52,360.

Section 10.3 You Try: The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding You Try problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same of course would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form:

Example 13: The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution: In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \times 2002 = 6,012,006$ possible ways of selecting the members of the Defense Subcommittee.

SECTION 10.4: FINDING PROBABILITIES USING COUNTING

Up to this point we have been using permutations and combinations to help us count, but we can also use permutations and combinations to help us answer more complex probability questions. In the following examples, we will use permutations and combinations to find probabilities of certain events.

Example 14: A 4 digit PIN number is selected. What is the probability that there are no repeated digits?

Solution: There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \times 10 \times 10 \times 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \times 9 \times 8 \times 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is

$$\frac{{}_{10}P_4}{10^4} = \frac{5040}{10,000} = 0.504$$

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 15: Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution: There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. In other words, the alternative to "at least one" is having **none**. In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{no shared Birthday}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday}) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082.$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 16: Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution: Continuing the pattern of the previous example, the answer should be

$$P(\text{shared Birthday}) = 1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{shared Birthday}) = 1 - \frac{{}^{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and which suggests a nice formula as we continue to expand the population of our group.

Example 17: Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution: Here we can calculate

$$P(\text{shared Birthday}) = 1 - \frac{{}^{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

Section 10.4A You Try: Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

Just as we did with permutations, we can now use combinations to move beyond **counting** and help us answer more complex **probability** questions.

Example 18: In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins \$1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution: In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12,271,512} \approx 0.000000081$$

Example 19: In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of \$1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution: As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \times {}_{42}C_1 = 6 \times 42 = 252$. So the probability of winning the second prize is.

$$\frac{{}_6C_5 \times {}_{42}C_1}{{}_{48}C_6} = \frac{252}{12,271,512} \approx 0.0000205$$

Example 20: Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution: In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses); in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \times {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{{}_4C_1 \times {}_{48}C_4}{{}_{52}C_5} = \frac{778,320}{2,598,960} \approx 0.299$$

Example 21: Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution: The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{{}_4C_2 \times {}_{48}C_3}{{}_{52}C_5} = \frac{103,776}{2,598,960} \approx 0.0399$$

Section 10.4B You Try: Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Chapter 10 – Answers to You Try Problems

10.1: 480 different schedules are possible

10.2A: 479,001,600

10.2B: a) $26^5 = 11,881,376$ b) ${}_{26}P_5 = 7,893,600$

10.3: 20,030,010

10.4A: 0.1169

10.4B: 0.000009234

CHAPTER 10 – PRACTICE PROBLEMS

- 1) A boy owns 5 pairs of pants, 4 shirts, 1 tie, and 3 jackets. How many different outfits can he wear to school if he must wear one of each item?
- 2) A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?
- 3) Bailey has 5 skirts, 9 blouses, and 10 pairs of shoes. How many different skirt—blouse—shoe outfits can she wear?
- 4) At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three—course meals can you have?
- 5) A pizza place has pizzas in 4 different sizes, with 12 different flavors for the crust, and 10 different toppings. How many one—topping pizzas can be ordered?
- 6) Standard automobile license plates in a country display 1 numbers, followed by 2 letters, followed by 4 numbers. How many different standard plates are possible in this system? (Assume repetitions of letters and numbers are allowed.)
- 7) All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles? (Assume repetitions of letters and numbers are allowed.)
- 8) A true false test contains 13 questions. In how many different ways can this test be completed? (Assume we don't care about our scores.)
- 9) A computer password must be eight characters long. How many passwords are possible if only the 26 lowercase letters of the alphabet are allowed? (Assume repetitions of letters are allowed.)
- 10) A computer password must be ten characters long. How many passwords are possible if numbers, 26 lowercase letters, and 26 uppercase letters are allowed? (Assume repetitions of numbers and letters are allowed.)
- 11) A gate has a standard keypad with the digits 0 through 9. How many possible code combinations are there if the code is 5 digits long. (Assume repetitions of numbers are allowed.)

College Math – Chapter 10 Practice Problems

- 12) You are taking a quiz that has 11 multiple-choice questions. If each question has 4 possible answers, how many different ways are there to answer the test?
- 13) How many three letter "words" can be made from 4 letters "FGHI" if
- a) repetition of letters is allowed
 - b) repetition of letters is not allowed
- 14) How many four-letter "words" can be made from 6 letters "AEBWDP" if
- a) repetition of letters is allowed
 - b) repetition of letters is not allowed
- 15) How many three letter "words" can be made from 10 letters "FGHIJKLMNO" if
- a) repetition of letters is allowed
 - b) repetition of letters is not allowed
- 16) A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?
- 17) In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants? (Assume a contestant can only win one prize.)
- 18) Seven Olympic sprinters are eligible to compete in the 4 x 100 meter relay race for the USA Olympic team. How many four person relay teams can be selected from among the seven athletes?
- 19) A computer user has downloaded 25 songs using an online file sharing program and wants to create a CD—R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD—R is important to him, how many different CD—Rs could he make from the 25 songs available to him?
- 20) In how many ways can 4 different pizza toppings be chosen from 12 available toppings?
- 21) At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded? (Assume no one can receive more than one prize).
- 22) In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?
- 23) In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?
- 24) How many ways can 5 different cards be dealt from a standard 52—card deck?

- 25) A jury pool consists of 18 people. How many different ways can 3 people be chosen to serve on a jury?
- 26) A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?
- 27) The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee (STMMS) consists of 8 Republicans and 7 Democrats. How many ways can members of the STMMS be chosen from the Committee?
- 28) What is the probability that if 6 letters are typed, no letters are repeated?
- 29) Four letters are typed, without repetition. What is the probability that all 4 will be vowels?
- 30) Five letters are typed, with repetition allowed. What is the probability that all 5 will be vowels?
- 31) You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?
- 32) A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.
- 33) In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, then the player wins second prize. What is the probability of winning this prize?
- 34) In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, then the player wins third prize. What is the probability of winning this prize?
- 35) Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.
- 36) Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.

CHAPTER 11 – DESCRIBING DATA

Chapter 11 Learning Objectives:

- 1) Use bar graphs, pie charts, and histograms, to graphically display data
- 2) Identify distribution shapes from histograms
- 3) Calculate and interpret the mean, median, and mode
- 4) Compute and interpret the range and standard deviation
- 5) Calculate and interpret the quartiles, 5 number summary, and boxplots

Section 11.1 – Presenting Categorical Data Graphically

Section 11.2 – Presenting Quantitative Data Graphically

Section 11.3 – Describing a Distribution

Section 11.4 – Measures of Central Tendency

Section 11.5 – Measures of Variation

Section 11.6 – Quartiles, Five Number Summary, and Boxplots

SECTION 11.1: PRESENTING CATEGORICAL DATA GRAPHICALLY

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. Categorical data are usually non-numeric (like eye color or political party affiliation). We usually begin working with categorical data by summarizing the data into a **frequency table**.

Frequency Table

A frequency table is a table with two columns. One column lists the possible values of the variable, and the other lists the corresponding frequency for each value (how many items fit into each category). Many times a 3rd column is added, called the relative frequency column. The relative frequency expresses the frequency of each possible value, relative to the whole. Relative frequencies can be written as fractions, decimals, or percentages.

Example 1: An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. Use the data summarized in the frequency table below to create a relative frequency table.

Color	Blue	Green	Red	White	Black	Grey
Frequency	25	52	41	36	39	23

Solution: When you add together the frequencies for each possible value of the variable, you end up with the total number of observed values. In this case, that total is 216. If we wanted to add a relative frequency column, we would find the relative frequency for each color by dividing the corresponding frequency by the total number of observations. The new relative frequency table would be as follows:

Color	Frequency	Relative Frequency
Blue	25	$\frac{25}{216} = 0.1157 = 11.57\%$
Green	52	$\frac{52}{216} = 0.2407 = 24.07\%$
Red	41	$\frac{41}{216} = 0.1898 = 18.98\%$
White	36	$\frac{36}{216} = 0.1667 = 16.67\%$
Black	39	$\frac{39}{216} = 0.1806 = 18.06\%$
Grey	23	$\frac{23}{216} = 0.1065 = 10.65\%$

Note: The relative frequency column should always sum to 1 (or 100%) since you are representing the entire data set.

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. In this section we will work with bar graphs that display categorical data; the next section will be devoted to histograms that display quantitative data.

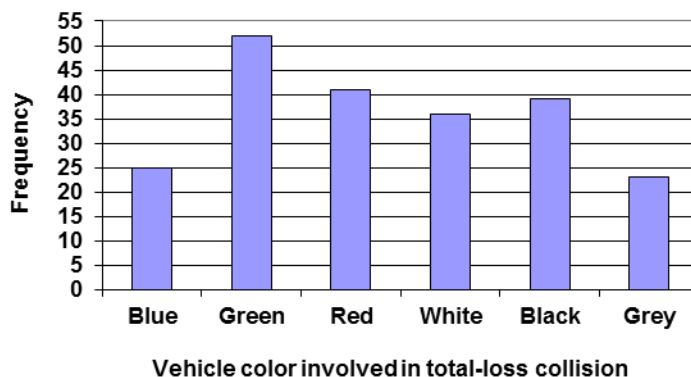
Bar Graphs

A **bar graph** is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

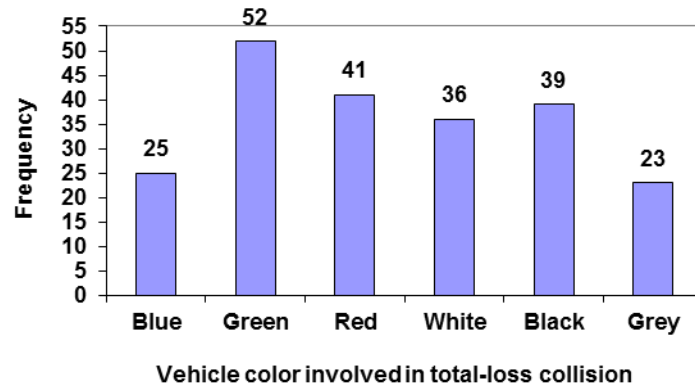
To construct a bar graph, we need to draw a vertical axis and a horizontal axis. The vertical direction will have a scale and measure the frequency of each category; the horizontal axis has no scale in this instance, and represents the possible values of the variable of interest. The construction of a bar chart is most easily described by use of an example.

Example 2: Make a bar graph from the car data in Example 1.

Solution: Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:



Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:



Section 11.1A You Try: Create a frequency table and bar graph to illustrate the grades on a history exam: A: 12 students, B: 19 students, C: 14 students, D: 4 students, F: 5 students

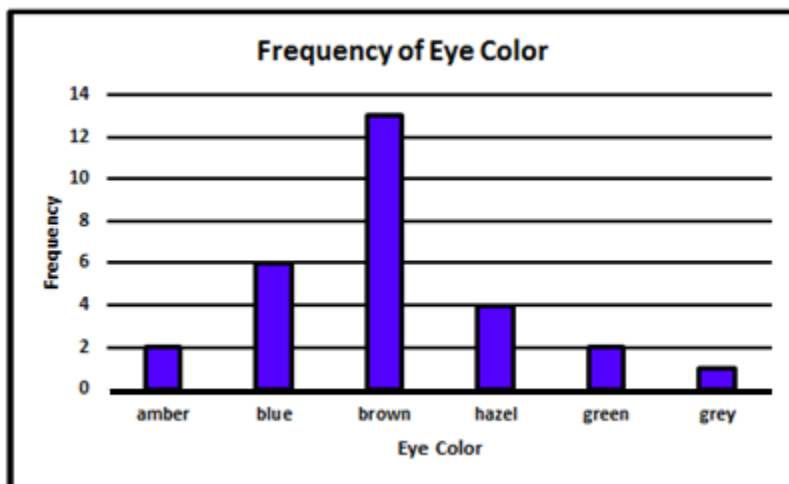
Example 3: Use the bar chart from example 2 to answer the following questions.

- How many of the vehicles were red?
- Which color of vehicle was involved in the most total-loss collisions?
- What percentage of the vehicles involved in total-loss collisions were blue?

Solution:

- The colors of the vehicles are listed along the horizontal axis, and the frequency (number) of vehicles is found on the vertical axis. First, find the column along the horizontal axis labeled “Red”. From there, we can observe that the bar corresponding to “Red” has a frequency of 41. There were 41 red vehicles.
- In order to find the color of vehicle that was involved in the most total-loss collisions, we need to find the tallest bar. The tallest bar will have the largest frequency. The tallest bar has a frequency of 52, and corresponds to the color “Green”. The vehicle color involved in the most total-loss collisions is Green.
- To find the percentage of blue vehicles, we need two things: the number of blue vehicles, and the total number of vehicles. We can see by looking at the height of the bar that corresponds with “Blue” that there were 25 blue vehicles. To find the total number of vehicles, we add the frequency from each bar: $25 + 52 + 41 + 36 + 39 + 23$. The total number of vehicles is 216. To find the percentage of blue vehicles divide $25/216$ to get 0.115740. Multiply by 100 to convert to a percentage. 11.57% of the vehicles involved in total-loss collisions were blue.

Section 11.1B You Try: The bar graph below represents the frequency of eye color in a class of students.



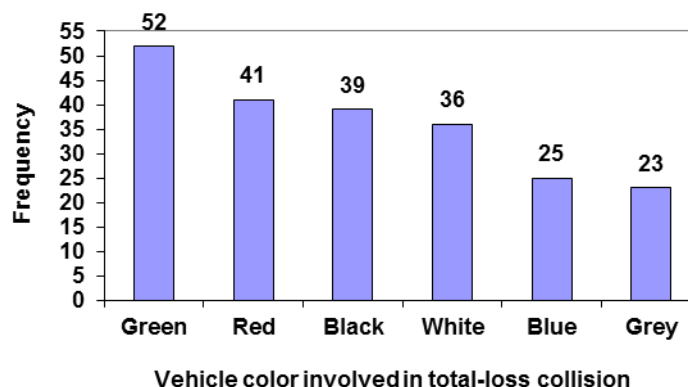
- How many students have hazel eyes?
- What eye color is most frequent in the class?
- What eye color is least frequent in the class?
- How many students are in the class?
- What percentage of the class has blue eyes? Round to the nearest hundredth.

In some cases, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart, even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart**.

A **Pareto chart** is a bar graph ordered from highest to lowest frequency

Example 4: Make a Pareto chart from the car data in Example 1.

Solution: Transforming our bar graph from earlier into a Pareto chart, we get:



Pie Chart

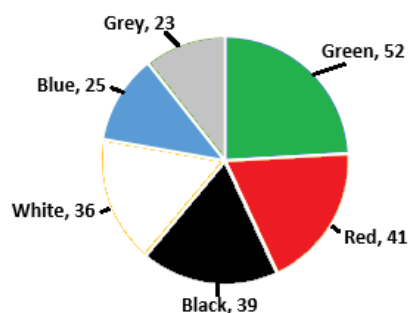
A **pie chart** is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories. These relative frequencies are usually expressed as a percentage.

Pie charts can often benefit from including frequencies or relative frequencies (percentages) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.

Example 5: Make a Pie chart from the car data in Example 1.

Solution: The corresponding pie chart is shown below.

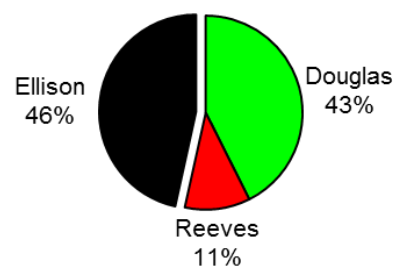
Vehicle color involved in total-loss collisions



Example 6: The pie chart to the right shows the percentage of voters supporting each candidate running for a local senate seat. If there are 20,000 voters in the district, what percent and number of voters support Reeves?

Solution: The pie chart shows 11% of voters support Reeves. Reeves is supported by 11% of 20,000 people which is 2200 people.

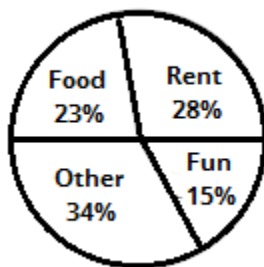
Voter preferences



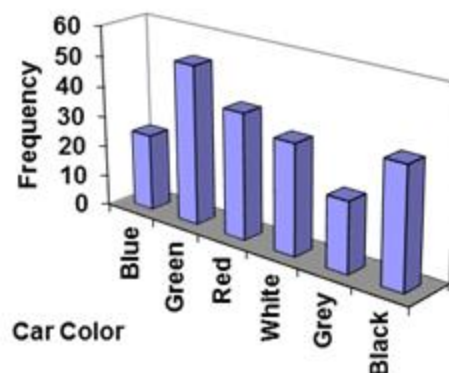
Pie charts look nice, but are harder to draw by hand than bar charts since to draw them accurately we would need to compute the angle each wedge cuts out of the circle, then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org, Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs⁹.

⁹ For example: <http://nces.ed.gov/nceskids/createAgraph/> or <http://docs.google.com>

Section 11.1C You Try: Logan categorized his spending for this month into four categories: Rent, Food, Fun, and Other. The percentages he spent in each category are pictured in the pie chart below. If Logan spent a total of \$2400 this month, how much did he spend on Food?

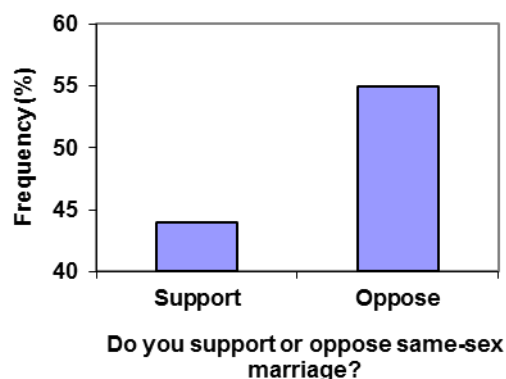
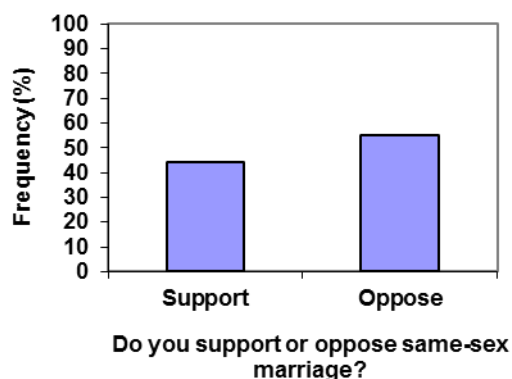


Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown to the right are usually not as effective as their two-dimensional counterparts



Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

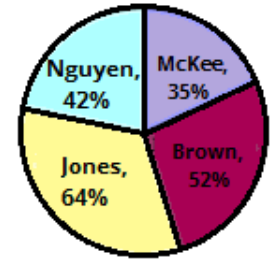
Example 7: Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December, 2008¹⁰.



Solution: The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like twice as many people oppose marriage rights as support it.

¹⁰CNN/Opinion Research Corporation Poll. Dec 19-21, 2008, from <http://www.pollingreport.com/civil.htm>

Section 11.1D You Try: A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart to the right present a good representation of this data? Explain.



SECTION 11.2: PRESENTING QUANTITATIVE DATA GRAPHICALLY

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 8: A teacher records scores on a 20-point quiz for the 30 students in his class. Create a frequency table.

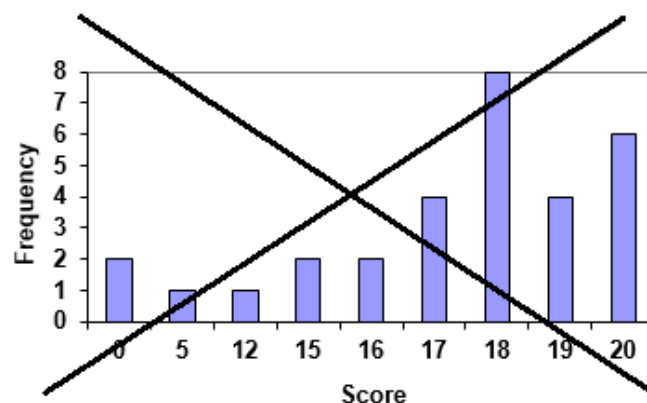
The scores are:

19 20 18 18 17 18 19 17 20 18 20 16 20 15 17 12 18 19 18 19 17 20 18 16 15 18 20 5 0 0

Solution: These scores could be summarized into a frequency table by grouping like values:

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it seems like we could create a standard bar chart from this summary, like we did for categorical data:



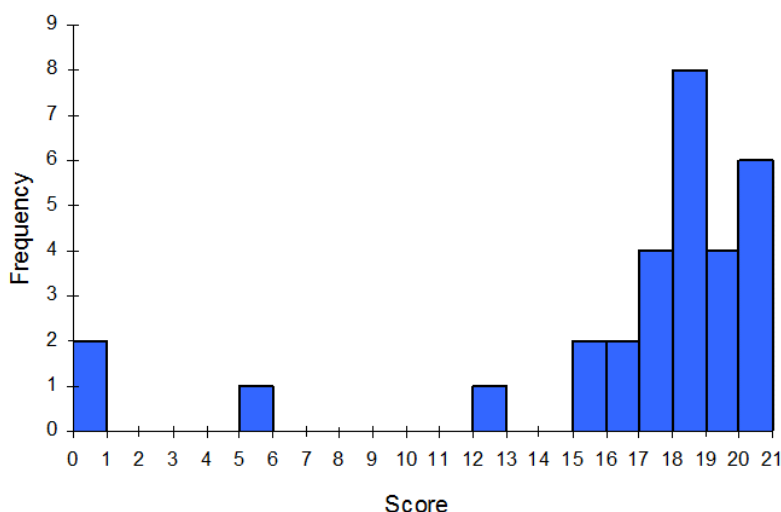
However, since the scores are numerical values, the bar graph above doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be more correct to treat the horizontal axis as a *number line*. This type of graph is called a **histogram**.

Histogram

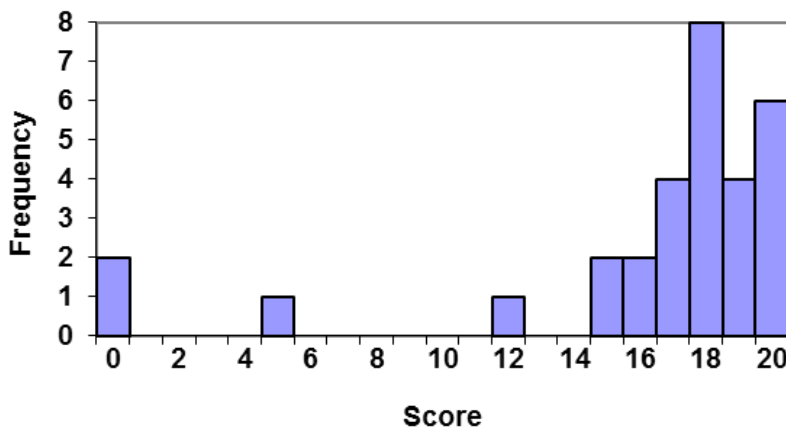
A **histogram** is like a bar graph, but where the horizontal axis is a number line. The vertical axis still represents frequency, and the horizontal axis still represents the variable of interest.

Example 9: For the quiz score values in example 8, draw a histogram.

Solution: The histogram for the quiz score data is below:



Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. Some people choose to have bars start at $\frac{1}{2}$ values to avoid this ambiguity as shown below.



Unfortunately, not a lot of common software packages can correctly graph a histogram.

Fortunately, your graphing calculator can quickly create an accurate histogram using the STAT PLOT option.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table, and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals**.

Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 10: Suppose that we have collected weights from 100 male subjects as part of a nutrition study.

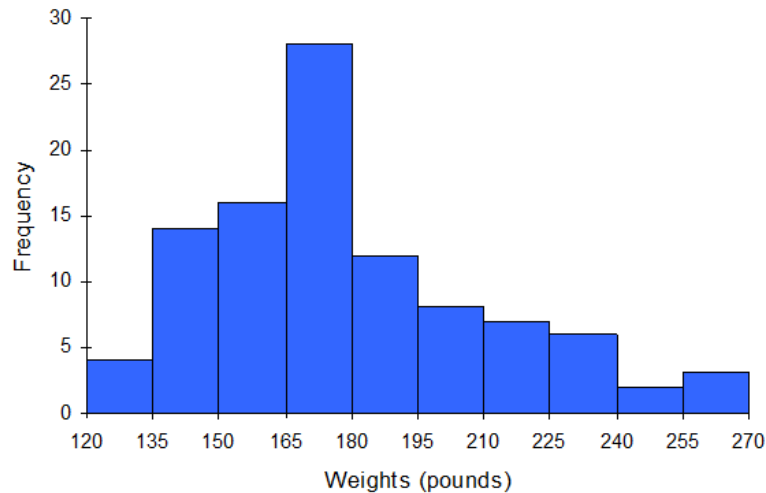
Interval	Frequency
120 - 134	4
135 - 149	14
150 - 164	16
165 - 179	28
180 - 194	12
195 - 209	8
210 - 224	7
225 - 239	6
240 - 254	2
255 - 269	3

Use the frequency table above to answer the following questions:

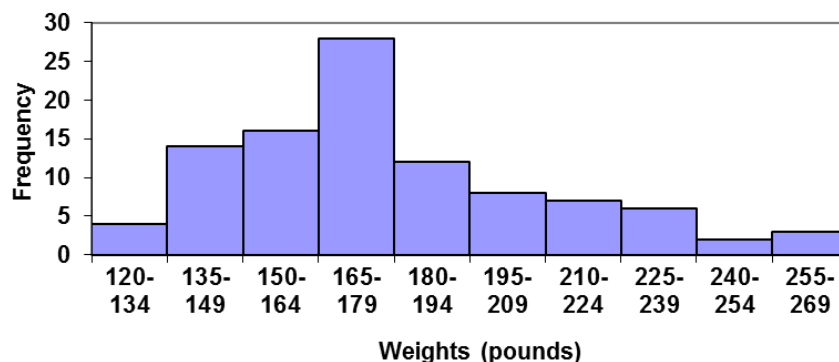
- Draw a histogram of this data.
- How many men had weights between 195 and 209 pounds?
- How many men had weights that were at least 225 pounds?
- What percent of men weighed less than 150 pounds?

Solution:

a) The histogram for the weights of these 100 men is shown below:



In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.



- b) There are 8 men with weights between 195 and 209 pounds.
- c) To find the number of men with weights that were at least 225 pounds, we consider which columns represent weights of 225 pounds or more. This is the last 3 columns in the histogram. The number of men with weights that were at least 225 pounds is $6 + 2 + 3 = 11$. There are 11 men that have weights that are at least 225 pounds.
- d) To find the percentage of men weighing less than 150 pounds, we first need to find the number of men weighing less than 150 pounds. This is represented by the first two bars in the histogram. There are $4 + 14 = 18$ men weighing less than 150 pounds. Next, we need the total number of men, which we are given as 100. The percentage of men weighing less than 150 pounds is $18/100 = .18 = 18\%$.

Section 11.2 You Try: The total cost of textbooks for the term was collected from 36 students.

\$140	\$160	\$160	\$165	\$180	\$220	\$235	\$240	\$250	\$260	\$280	\$285
\$285	\$285	\$290	\$300	\$300	\$305	\$310	\$310	\$315	\$315	\$320	\$320
\$330	\$340	\$345	\$350	\$355	\$360	\$360	\$380	\$395	\$420	\$460	\$460

- Create a histogram for this data.
- How many students had textbook costs less than \$200?
- What percentage of students had textbook costs over \$400?

Calculator Instructions for Drawing a Histogram Using a TI-83/84:

- Turn on the calculator
- Press the "STAT" key
- Hit "Enter" on option 1: "Edit"

This will bring you to a screen that contains lists: L1, L2, L3, etc.

- Enter the data values (one value per row) into L1. For any negative values you need to use the (-) key, not the subtraction key. Continue until all data is entered into L1.
- Press the 2nd key and then "STAT PLOT" (top left of calculator)
- Press "Enter" on 1: "Plot 1"
- Use the arrows to "ON" to turn on Plot 1. Hit "Enter".
- Use the arrows to go to the histogram picture and hit "Enter".
- Make sure that Xlist displays L1
- Press the 2nd key and then "STAT PLOT" again.

Make sure to turn off all other stat plots, and clear all equations before graphing the histogram.

- Press the ZOOM menu.
- Scroll down to option 9: "ZoomStat" and hit "Enter".

SECTION 11.3: DESCRIBING A DISTRIBUTION

Now that we have learned how to display data graphically, we will begin to discuss how to describe the important characteristics of a particular variable of interest. Of particular importance is the variable's distribution.

Distribution

The **distribution** of a variable tells you:

- 1) all possible values the variable can take on, and
- 2) the frequency that each of the possible values occurs.

In the last sections, we looked at frequency tables and histograms, which are two ways of showing a variable's distribution. Distributions can also be shown by fitting smooth curves to a graphical display of a data set, or using a formula.

In statistics, when we consider differences between distributions, we usually look at several characteristics of the distribution:

Shape
Outliers
Center
Spread

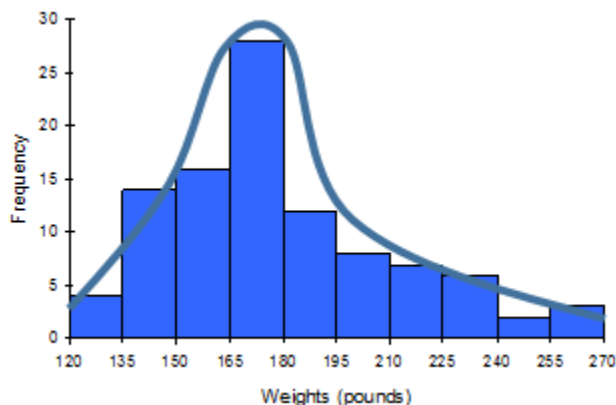
Together we refer to these as the "SOCS" of the distribution

Distribution Shapes

The shape of a distribution is frequently estimated by drawing a smooth curve to a histogram, as shown in the following diagram:

In the histogram below, a smooth curve has been drawn to roughly approximate the shape of the distribution for the variable "Weights".

This particular distribution is right-skewed.



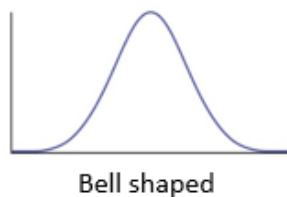
The previous example showed just one of the many possible distribution shapes that can be observed. There are several common distribution shapes, each with unique combinations of number of peaks, symmetry, center, and spread.

The most commonly used distribution curve in elementary statistics is a bell-shaped distribution curve, but it is important to realize that there are many possible distribution shapes. Some are symmetric, some are right-skewed, left-skewed, unimodal (one peak), or bimodal (two peaks). A few of the most common shapes are shown below.

Symmetric distribution

A **symmetric distribution** is a distribution shape in which the left and right sides of the distribution are roughly mirror images of one another.

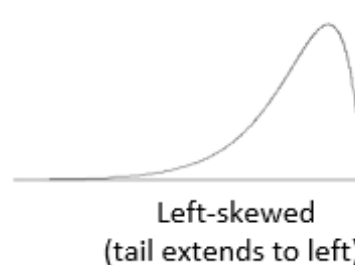
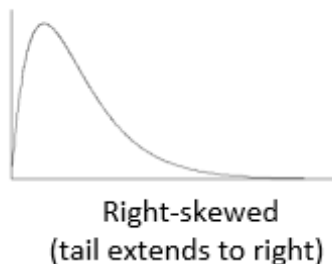
Examples of Symmetric Distribution Shapes:



Skewed distribution

A **skewed distribution** is a distribution shape in which the data is asymmetrical, and tends to cluster toward one side, while having a longer “tail” on the other side.

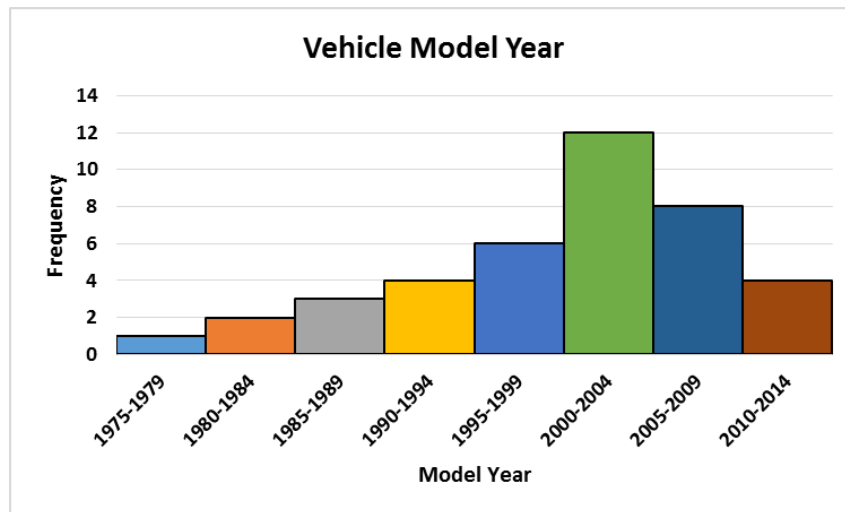
Examples of Skewed Distribution Shapes:



Note: In practice, there will be variables whose distribution shapes do not exactly fit any of the shapes described above.

Example 11: The histogram below shows the “Model Year” for a sample of 40 vehicles.

- a) Is the distribution symmetric or skewed?
b) Classify the shape of the distribution as bell shaped, uniform, right-skewed, or left-skewed

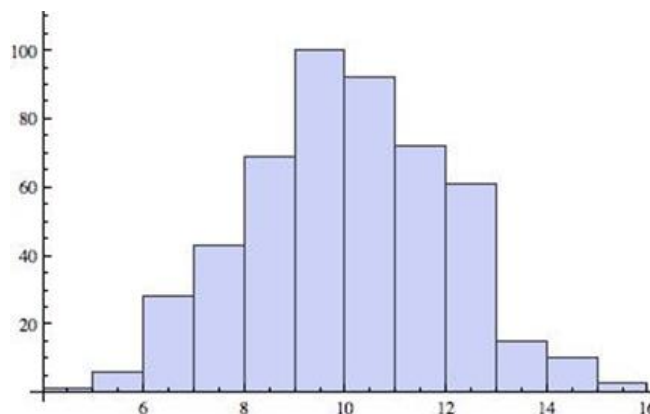


Solution:

- a) The distribution is a skewed distribution, as the data clusters on the right side and has a longer tail on the left side.
b) This distribution is left-skewed, as the longer tail extends to the left.

Example 12: Given the histogram below, answer the following questions.

- a) Is the distribution symmetric or skewed?
b) Classify the shape of the distribution as bell shaped, uniform, right-skewed, or left-skewed.



Solution:

- a) The distribution is a symmetric distribution, as the data clusters fairly evenly on both the left and right sides.
b) This distribution is bell-shaped.

Outliers

An **outlier** is a data value that is “extreme” when compared to the rest of the values in the data set. It may be extreme in either direction, meaning it may be very small when compared to the rest of the values in the data set, or it may be very large when compared to the rest of the values in the data set.

While there are several formal methods for determining if a data value is an outlier (or a potential outlier), the determination of outliers can be somewhat subjective. This means that it can be left up to the researcher to determine which observations are “extreme”.

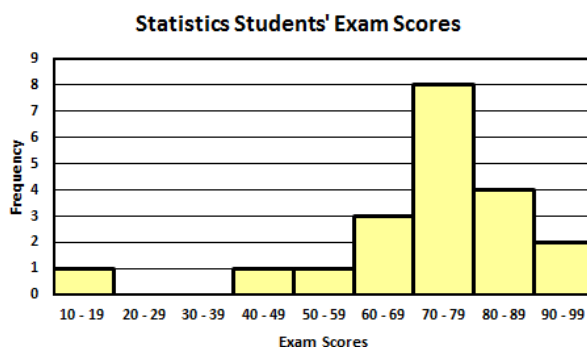
Example 13: Consider the data set below which consists of “Exam Scores” for a sample of 20 Statistics students:

89	73	76	95	72	61	70	18	65	57
77	93	85	81	72	48	69	75	83	70

Do there appear to be any outliers in the data set? Why or why not?

Solution: It can be difficult to look at a list of data and try to identify any “extreme” or unusual values. These outliers are easier to identify by looking at a graph of the data.

Create a histogram, and look for any outliers. They will be values that appear to fall far outside the normal values.



Notice the graph shows one bar that is “far” below the other bars in the graph. When looking more closely, we see that this bar represents 1 test score. If we look back at the data set, we can see this bar represents 1 student who scored an 18 on the Exam.

The value 18 appears to be an outlier, because it is “far” outside the original group of data values. It is important to recognize that there is an outlier in the data set, as the outlier may affect the descriptive measures of the mean and standard deviation (coming up in the next sections).

Since outliers can affect the descriptive measures we use to describe a data set (specifically the mean as a measure of center and the standard deviation as a measure of spread), it is important to visually inspect a data set to see if there appear to be any outliers. We will come back to this idea in the next several sections.

Recall, we started this chapter by defining a distribution, and describing the “SOCS” (Shape, Outliers, Center, and Spread) of a distribution. Now that we have spent time introducing the “Shape” of a distribution, and briefly discussing “Outliers”, we will move on to measuring the “Center”. We will discuss “Spread” later on in Sections 11.3 and 11.4.

SECTION 11.4: MEASURES OF CENTRAL TENDENCY

The “Center” is an important aspect of a distribution. When we refer to the “Center” we are trying to get an idea of the location of the middle of the data set. There are three common measures of central tendency. They are the mean, median, and mode.

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the **arithmetic mean**, or just plain old **mean** for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average".

Mean

The **mean** of a set of data is the sum of the data values divided by the number of values.

Example 14: Marci's exam scores for her last math class were: 79, 86, 82, 94. Find the mean.

Solution: The mean of these values would be: $\frac{79+86+82+94}{4} = 85.25$

Typically we round means to one more decimal place than the original data had. In this case, we would round 85.25 to 85.3.

Example 15: The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below. Find the mean.

37 33 33 32 29 28 28 23 22 22 22 21 21 21 20
20 19 19 18 18 18 18 16 15 14 14 14 12 12 9 6

Solution: Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get $634/31 = 20.4516$. It would be appropriate to round this to 20.5.

It would be most correct for us to report that “The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes,” but it is not uncommon to see the more casual word “average” used in place of “mean.”

Section 11.4A You Try: The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the mean price.

Example 16: The one hundred families in a particular neighborhood are asked their annual household income, to the nearest \$5 thousand dollars. The results are summarized in a frequency table below. Find the mean.

Income (\$1000's)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution: Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15 + \cdots + 15}^{6 \text{ terms}} + \overbrace{20 + \cdots + 20}^{8 \text{ terms}} + \overbrace{25 + \cdots + 25}^{11 \text{ terms}} + \cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as $15 \times 6 = 90$. Using this simplification, we get

$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7}{100} = \frac{3390}{100} = 33.9$$

The mean household income of our sample is 33.9 thousand dollars (\$33,900).

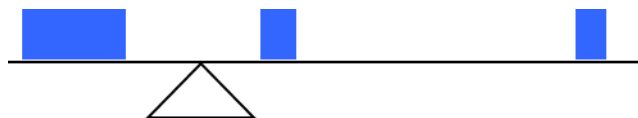
Example 17: Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of \$5 million (\$5000 thousand). Adding this to our sample, find the mean.

Solution: Adding \$5 million to our sample, our mean is now:

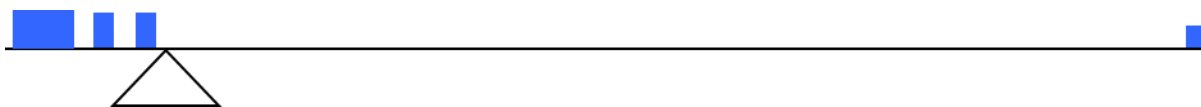
$$\frac{15 \cdot 6 + 20 \cdot 8 + 25 \cdot 11 + 30 \cdot 17 + 35 \cdot 19 + 40 \cdot 20 + 45 \cdot 12 + 50 \cdot 7 + 5000 \cdot 1}{101} = \frac{8390}{101} = 83.069$$

While 83.1 thousand dollars (\$83,069) is the correct mean household income, it no longer represents a “typical” value.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



If we graph our household data, the \$5 million data value is so far out to the right that the mean has to adjust up to keep things in balance



For this reason, when working with data that have **outliers** – as described in the previous section - it is common to use a different measure of center, the **median**.

Median

The **median** of a set of data is the value in the middle when the data is in order. To find the median, **begin by listing the data in order from smallest to largest**.

If the number of data values, n , is odd, then the median is the middle data value. The position in the ordered list of this value can be found using the formula: $\frac{n+1}{2}$

Note: this formula gives you the POSITION of the median, not the value of the median itself.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values. The position of the median is still $\frac{n+1}{2}$ but this will not be a whole number.

The median will be found by taking the mean of the numbers in the positions immediately below and immediately above the position found using the formula.

Example 18: Returning to the football touchdown data, we would start by listing the data in order from smallest to largest.

6 9 12 12 14 14 14 15 16 18 18 18 18 19 19 20
20 21 21 21 22 22 22 23 28 28 29 32 33 33 37

Solution: Since there are 31 data values, an odd number, the median will be the middle number, in the $(31 + 1) / 2 = 32 / 2 = 16^{\text{th}}$ position (leaving 15 values below and 15 above). The 16^{th} data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 19: Find the median of these quiz scores: 15 20 18 16 14 18 12 15 17 17

Solution: We start by listing the data in order: 12 14 15 15 16 17 17 18 18 20. Since there are 10 data values, an even number, there is no one middle number. To find the median, use the formula to find the position of the median with $n = 10$, $(10 + 1) / 2 = 11 / 2 = 5.5$, so we need the mean of the 5^{th} and 6^{th} numbers in the ordered list. The 5^{th} number in the ordered list is 16, and the 6^{th} number in the ordered list is 17. So we find the mean of the two middle numbers, 16 and 17, and get $(16 + 17) / 2 = 16.5$.

The median quiz score was 16.5.

Section 11.4B You Try: The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the median price.

Example 20: Let us return now to our original household income data. Find the median.

Income (\$1000's)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution: Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, using the position formula we get $(100 + 1) / 2 = 101 / 2 = 50.5$. We need to find the mean of the middle two data values - the 50^{th} and 51^{st} data values. To find these, we start counting up from the bottom:

There are 6 data values of \$15, so $\rightarrow$ Values 1 to 6 are \$15 thousand

The next 8 data values are \$20, so $\rightarrow$ Values 7 to $(6+8) = 14$ are \$20 thousand

The next 11 data values are \$25, so $\rightarrow$ Values 15 to $(14+11) = 25$ are \$25 thousand

The next 17 data values are \$30, so $\rightarrow$ Values 26 to $(25+17) = 42$ are \$30 thousand

The next 19 data values are \$35, so $\rightarrow$ Values 43 to $(42+19) = 61$ are \$35 thousand

From this we can tell that values 50 and 51 will be \$35 thousand, and the mean of these two values is \$35 thousand. The median income in this neighborhood is \$35 thousand.

Example 21: Add the \$5 million to the previous data set and find the median.

Solution: If we add in the new neighbor with a \$5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is \$35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode**.

Mode

The **mode** is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 22: In our vehicle color survey, we collected the data below. Find the mode.

Color	Blue	Green	Red	White	Black	Grey
Frequency	25	52	41	36	39	23

Solution: For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode if several categories have the same frequency, or no modes if each every category occurs only once.

Section 11.4C You Try: Reviewers were asked to rate a product on a scale of 1 to 5. The data are given below. Find the mean, median, and mode of the data.

Rating	1	2	3	4	5
Frequency	4	8	7	3	1

SECTION 11.5: MEASURES OF VARIATION

Another component of describing a data set is how much “Spread” there is in the data set. In other words, how much the data in the distribution vary from one another. It may seem like once we know the center of a data set, we know everything there is to know. The first example will demonstrate why we need measures of variation (or spread).

There are several ways to measure this “Spread” of the data. The three most common measures are the range, standard deviation, and quartiles. In this section we will learn about the range and standard deviation. We will discuss quartiles in the following section.

We will focus first on the simplest measure of spread, called the **range**.

Range

The **range** is the difference between the maximum value and the minimum value of the data set.

Example 23: Consider these three sets of quiz scores. Find the range of each.

Section A: 5 5 5 5 5 5 5 5 5

Section B: 0 0 0 0 0 10 10 10 10 10

Section C: 4 4 4 5 5 5 5 6 6 6

Solution: All three of these sets of data have a mean of 5 and median of 5. If we only calculated a measure of center for each set of scores, we would say the three sets are all identical, yet the sets of scores are clearly quite different. Calculating a measure of variability (or spread) will help identify how they are different.

For section A, the range is 0 since both maximum and minimum are 5 and $5 - 5 = 0$

For section B, the range is 10 since $10 - 0 = 10$

For section C, the range is 2 since $6 - 4 = 2$

Section 11.5A You Try: The price of a jar of peanut butter at 5 stores was: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the range of the prices.

In example 23, the range seems to be revealing how spread out the data is. However, suppose we add a fourth section, Section D: 0 5 5 5 5 5 5 5 10

This section also has a mean and median of 5. The range is 10, yet this data set is quite different than Section B. To better illuminate the differences, we’ll have to turn to more sophisticated measures of variation.

Standard Deviation

The **standard deviation** is a measure of variation based on measuring how far, on average, each data value deviates, or is different, from the mean. A few important characteristics:

- 1) Standard deviation is always positive. Standard deviation will be zero if all the data values are equal, and will get larger as the data spreads out.
- 2) Standard deviation has the same units as the original data.
- 3) Standard deviation, like the mean, can be highly influenced by outliers.

Using the data from Section D: 0 5 5 5 5 5 5 5 10, we could compute for each data value the difference between the data value and the mean. This will give us an idea of “how far” each value in the data set lies away from the mean.

data value	deviation: data value – mean
0	$0 - 5 = -5$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
5	$5 - 5 = 0$
10	$10 - 5 = 5$

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column the negative and positive values cancel each other out (this always happens), so instead we square every value in the second column:

data value	deviation: data value – mean	deviation squared
0	$0 - 5 = -5$	$(-5)^2$
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
5	$5 - 5 = 0$	0^2
10	$10 - 5 = 5$	5^2

We then add the squared deviations up to get $25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50$.

Ordinarily we would then divide by the number of scores, n , (in this case, 10) to find the mean of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n - 1$ (in this case, $10 - 1 = 9$).¹¹

So in our example, we would have $50/10 = 5$ if section D represents a population and $50/9 =$ about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

Variance can be a useful statistical concept, but note that the units of variance in this instance would be points – squared since we squared all of the deviations. What are points – squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

$$\text{Population standard deviation} = \sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$$

$$\text{Sample standard deviation} = \sqrt{\frac{50}{9}} \approx 2.4$$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data, as we have done above.

To Compute the Standard Deviation

- 1) Find the deviation of each data from the mean. In other words, subtract the mean from the data value.
- 2) Square each deviation.
- 3) Add the squared deviations.
- 4) Divide by n , the number of data values, if the data represents a whole population; divide by $n - 1$ if the data is from a sample.
- 5) Compute the square root of the result.

Example 24: Computing the standard deviation for Section B above.

Solution: We first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

¹¹ The reason we do this is highly technical, but we can see how it might be useful by considering the case of a small sample from a population that contains an outlier, which would increase the average deviation: the outlier very likely won't be included in the sample, so the mean deviation of the sample would underestimate the mean deviation of the population; thus we divide by a slightly smaller number to get a slightly bigger average deviation.

data value	deviation: data value – mean	deviation squared
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
0	$0 - 5 = -5$	$(-5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$
10	$10 - 5 = 5$	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25 + 25 + 25 + 25 + 25 + 25 + 25 + 25 + 25 + 25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are:

Section A: 5 5 5 5 5 5 5 5 5	Standard deviation: 0
Section B: 0 0 0 0 0 10 10 10 10 10	Standard deviation: 5
Section C: 4 4 4 5 5 5 5 6 6 6	Standard deviation: 0.8
Section D: 0 5 5 5 5 5 5 5 5 10	Standard deviation: 2.2

Section 11.5B You Try: The price of a jar of peanut butter at 5 stores were: \$3.29, \$3.59, \$3.79, \$3.75, and \$3.99. Find the standard deviation of the prices.

SEE THE END OF THE NEXT SECTION FOR INSTRUCTIONS ON FINDING SUMMARY STATISTICS USING THE TI – 83/84 CALCULATOR.

SECTION 11.6: QUARTILES, FIVE NUMBER SUMMARY, AND BOXPLOTS

The final measure of variability we must consider are the quartiles. Whereas the standard deviation was a measure of spread based around the mean, the quartiles are a measure of spread based around the median.

Quartiles

Quartiles are values that divide the data into quarters.

The first quartile (Q_1) is the value so that 25% of the data values are below it; the third quartile (Q_3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile (Q_2) is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q_1 , 25% is between Q_1 and Q_2 (the median), 25% is between Q_2 and Q_3 , and 25% is between Q_3 and the maximum value.

While quartiles are not a 1 – number summary of variation like the range or standard deviation, the quartiles are used with the median, minimum, and maximum values to form a **5 number summary** of the data.

Five Number Summary

The five number summary takes this form:

Minimum, Q_1 , Median (Q_2), Q_3 , Maximum

To find the quartiles, we first start by finding the second quartile, Q_2 . Recall, Q_2 is the same as the median since 50% of the data values are below Q_2 and 50% of the data values are above it.

To find the quartiles:

- 1) Begin by ordering the data from smallest to largest
- 2) Find the position of Q_2 (the median) using the formula $\frac{n+1}{2}$
- 3) Find the value of Q_2 (the median).
- 4) The median now divides your data set in half (if the original data set had an odd number of observations, exclude the median from both halves of your ordered list when you find Q_1 and Q_3).
- 5) To find Q_1 simply find the median of the lower half of the ordered list.
- 6) To find Q_3 simply find the median of the upper half of the ordered list.

Example 25: Suppose we had measured 9 females and their heights (in inches), sorted from smallest to largest are: 59 60 62 64 66 67 69 70 72. Give the five – number summary for this data set.

Solution: For this sample, the minimum value is 59, and the maximum is 72.

To find the **second quartile, the median**, we use the position formula $(n + 1)/2 = (9 + 1)/2 = 10/2 = 5$. The second quartile (median) will be the fifth data value: 66 inches.

Since the number of observations in the original data set was odd ($n = 9$), we exclude the median from both halves of the list to create:

Lower Half = 59, 60, 62, 64

Upper Half = 67, 69, 70, 72

To find the **first quartile** we find the median of the lower half of the ordered list. That list has 4 values, so $n = 4$, $(4 + 1)/2 = 5/2 = 2.5$. The first quartile will be the mean of the second and third data value in the lower half of the list: $(60 + 62)/2 = 61$ inches.

To find the **third quartile**, we find the median of the upper half of the ordered list. Since the upper half also has 4 values, this is still the mean of the 2nd and 3rd positions. The third quartile will be the mean $(69 + 70)/2 = 69.5$ inches.

The 5 number summary is: 59, 61, 66, 69.5, 72.

Example 26: Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are: 59 60 62 64 66 67 69 70. Give the five – number summary for this data set.

Solution: For this sample, the minimum value is 59, and the maximum is 70.

To find the **second quartile, the median**, we use the position formula $(n + 1)/2 = (8 + 1)/2 = 9/2 = 4.5$. The second quartile (median) will be the mean of the 4th and 5th numbers in the ordered list: 64 and 66. The second quartile (median) is $(64 + 66)/2 = 65$ inches.

Since the number of observations in the original data set was even ($n = 8$), we divide the list exactly in half to create:

Lower Half = 59, 60, 62, 64

Upper Half = 66, 67, 69, 70

To find the **first quartile** we find the median of the lower half of the ordered list. That list has 4 values, so $n = 4$, $(4 + 1)/2 = 5/2 = 2.5$. The first quartile will be the mean of the second and third data values in the lower half of the list: 60 and 62. The first quartile is $(60 + 62)/2 = 61$ inches.

To find the **third quartile**, we find the median of the upper half of the ordered list. Since the upper half also has 4 values, this is the mean of the second and third data values in the upper half of the list: 67 and 69. The third quartile is $(67 + 69)/2 = 68$ inches.

The 5 number summary for this data set would be: 59, 61, 65, 68, 70.

Section 11.6A You Try: The total cost of textbooks for the term was collected from 36 students. Find the 5 number summary of this data.

\$140	\$160	\$160	\$165	\$180	\$220	\$235	\$240	\$250	\$260	\$280	\$285
\$285	\$285	\$290	\$300	\$300	\$305	\$310	\$310	\$315	\$315	\$320	\$320
\$330	\$340	\$345	\$350	\$355	\$360	\$360	\$380	\$395	\$420	\$460	\$460

Note that the 5 number summary divides the data into four intervals, each of which will contain about 25% of the data. That does NOT mean that each of the four intervals will have the same width, just that each of the four intervals will contain 25% of the data.

It is often difficult to picture how the 5 number summary shows the variability in a data set. For visualizing data, there is a graphical representation of the 5 – number summary called a **box plot**, or box and whisker graph.

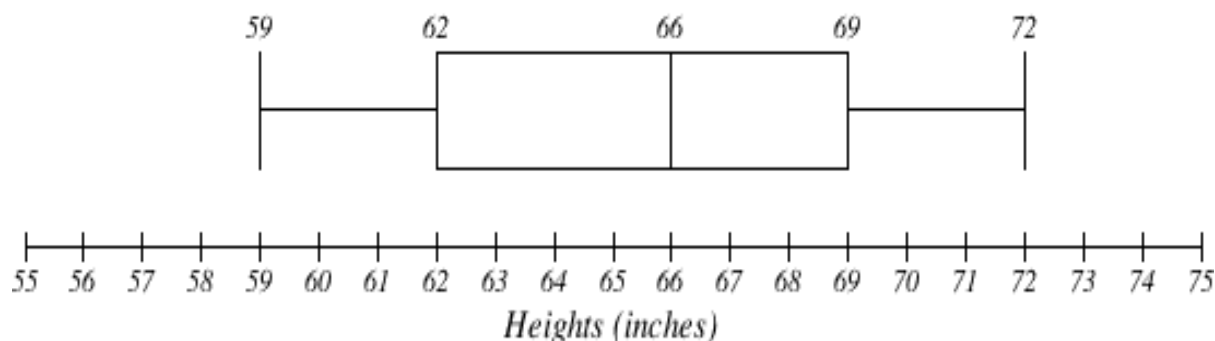
Boxplot (Box and Whisker Plot)

A **boxplot** is a graphical representation of a five – number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Note: It is important to use consistent intervals of values on the number line in a boxplot. For example, if you start your number line at 0, you may want to make tick marks at 10, 20, 30, etc. You should never use different intervals on the same axis. For example, do not have your first tick mark at 10, your next at 15, then the next at 35, etc. This will counteract the entire purpose of a boxplot, which is to see how the spread differs within the data set.

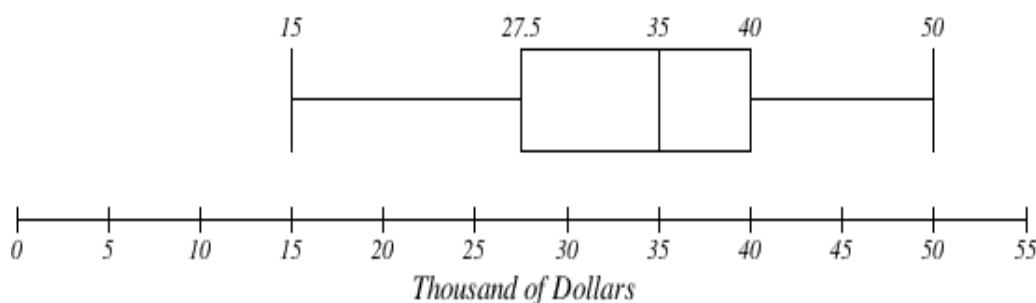
Example 27: The box plot below is based on the 9 female height data with 5 number summary: 59, 62, 66, 69, 72.



Notice that the horizontal axis used consistently spaced units (increments of 1).

Example 28: The box plot below is based on the household income data with 5 number summary: 15, 27.5, 35, 40, 50

Household incomes

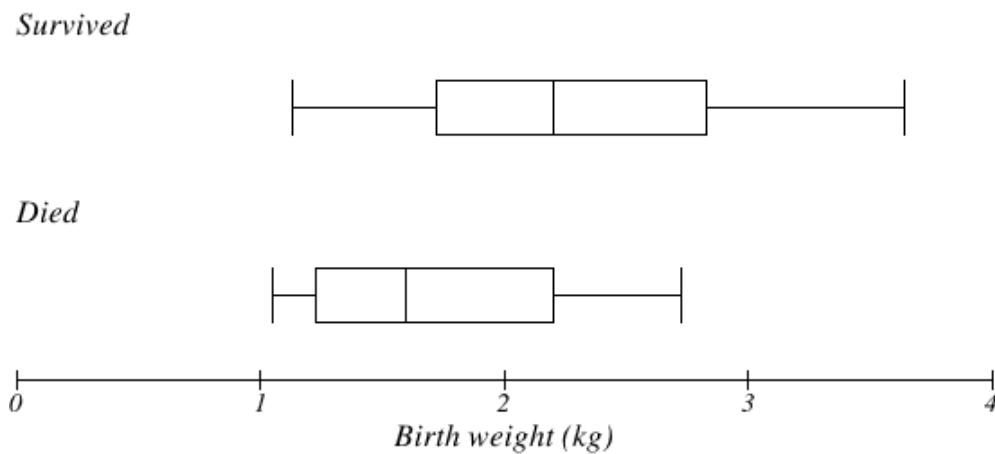


Notice that the horizontal axis used consistently spaced units (increments of 5). This is an example where you can quickly see that each of the 4 segments of the boxplot are NOT of equal width. For example, the first 25% of the data values have a spread of 12.5 thousand dollars (15 to 27.5) while the third 25% of the data values have a spread of only 5 thousand dollars (35 to 40). This just means that the 25% of incomes between the median and the third quartile are more tightly clustered together than the 25% of incomes between the minimum and the first quartile.

Section 11.6B You Try: Create a boxplot based on the textbook price data from the last You Try problem.

Box plots are particularly useful for comparing data from two populations.

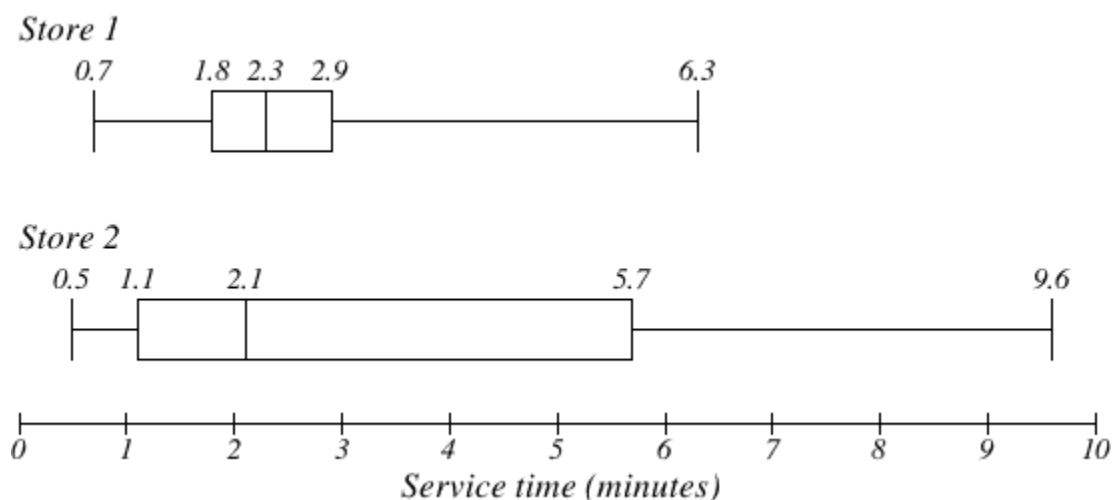
Example 29: The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS)¹². The boxplot is separated to show the birth weights of infants who survived and those that did not. Compare the boxplots.



Solution: Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Example 30: The box plot of service times for two fast food restaurants is shown below. Compare the boxplots.



¹² van Vliet, P.K. and Gupta, J.M. (1973) Sodium bicarbonate in idiopathic respiratory distress syndrome. *Arch. Disease in Childhood*, 48, 249–255. As quoted on <http://openlearn.open.ac.uk/mod/oucontent/view.php?id=398296§ion=1.1.3>

Solution: While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry? That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Percentiles

Percentiles are used in statistics to indicate a value below which a certain percentage of the data values fall. For example, if you score in the 60th percentile on a standardized test, it means that 60% of the other scores were lower than yours, (and 40% were higher).

Example 31: Answer the question below on percentiles.

- What percentage of the values in a data set lie at or below the 30th percentile?
- What percentage of the values in a data set lie at or above the 30th percentile?
- If 500 measurements were taken, approximately **how many** would be at or below the 30th percentile?
- If 500 measurements were taken, approximately **how many** would be at or above the 30th percentile?

Solutions:

- By definition, 30% of the data values lie at or below the 30th percentile.
- Since 30% lie below the 30th percentile, $100\% - 30\% = 70\%$ of the data values lie above it.
- $30\% \text{ of } 500 = .30(500) = 150$ values lie below the 30th percentile
- $70\% \text{ of } 500 = .70(500) = 350$ values lie above the 30th percentile

In general, for any set of numerical data, 50% of the data values are below the median, so the median will always represent the 50th percentile. The lower quartile (Q1) would be the 25th percentile, and the upper quartile (Q3) would be the 75th percentile.

Example 32: In Example 26 we looked at the heights (in inches), of a sample of 8 females. The heights were: 59 60 62 64 66 67 69 70, and we found the 5 number summary to be: 59, 61, 65, 68, 70. Find the 25th percentile and 75th percentile of the heights of females.

Solutions:

- We know that the 25th percentile is the height such that 25% of heights are lower, and 75% of heights are higher. This is the same as the first quartile. The 25th percentile is 61 inches.
- We know that the 75th percentile is the height such that 75% of heights are lower, and 25% of heights are higher. This is the same as the third quartile. The 75th percentile is 68 inches.

In this chapter, we only examine the process of finding the 25th, 50th, and 75th percentiles. Using other statistical techniques it is possible to find the data value corresponding to any percentile in the data set.

Calculator Instructions for Finding Summary Statistics Using TI – 83/84:

- 1) Turn on the calculator
- 2) Press the “STAT” key
- 3) Hit “Enter” on option 1: “Edit”

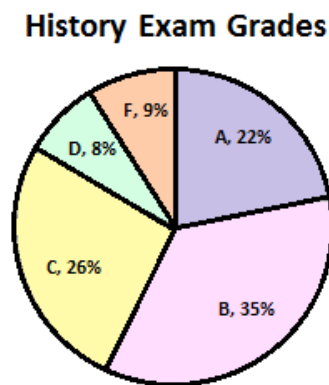
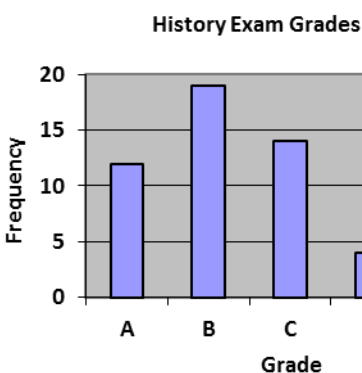
This will bring you to a screen that contains lists: L1, L2, L3, etc.

- 4) Enter the data values (one value per row) into L1. For any negative values you need to use the (–) key, not the subtraction key. Continue until all data is entered into L1.
- 5) Press the “STAT” key again
- 6) Use the arrow key to scroll over to “CALC”.
- 7) Select option 1: “1 – Var Stats”
- 8) Indicate that the data is in L1
- 9) Scroll down to “Calculate” and hit “Enter”

The summary statistics should now be displayed. You may scroll down with your arrow key to get remaining statistics. Using “1 – Var Stats” you can get the sample mean, sample standard deviation, population standard deviation, and 5 number summary.

Chapter 11 – Answers to You Try Problems

11.1A:



11.1B: a) 4 b) brown c) grey d) 28 e) 21.43%

11.1C: $\$2400(0.23) = \552

11.1D: While the pie chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percentages on a pie chart represent the percentage of the pie that the slice represents, and sums to 100%.

11.2: a) Answers vary b) 5 c) 8.33%

11.4A: Adding the prices and dividing by 5 we get the mean price: \$3.682

11.4B: First we put the data in order: \$3.29, \$3.59, \$3.75, \$3.79, \$3.99. Since there are an odd number of data values, the median will be the middle value, \$3.75.

11.4C: There are 23 ratings.

- a) The mean is 2.5
- b) There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
- c) The mode is the most frequent rating. The mode rating is 2.

11.5A: The range of the data is \$0.70.

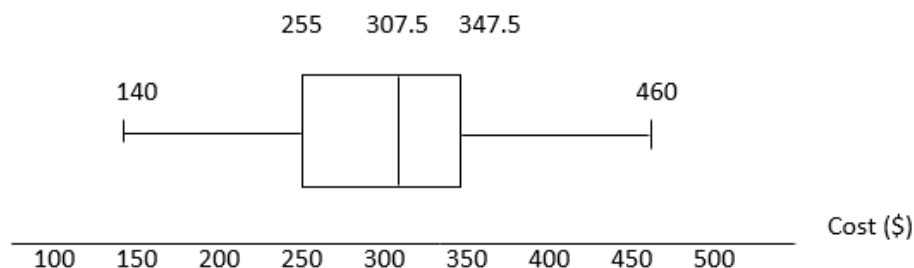
11.5B: The standard deviation of the data is \$0.26.

11.6A: The data is already in order, so we don't need to sort it first.

The minimum value is \$140 and the maximum is \$460.

- 1) There are 36 data values so $n = 36$. $(36 + 1)/2 = 37/2 = 18.5$, so the median is the mean of the 18th and 19th data values, \$305 and \$310. The median is \$307.50.
- 2) To find the first quartile, we calculate the median of the lower half of the ordered list. $Q1 = \$255$.
- 3) To find the third quartile, we calculate median of the upper half of the ordered list. $Q3 = \$347.50$.
- 4) The 5 number summary of this data is: \$140, \$255, \$307.50, \$347.50, \$460

11.6B:



CHAPTER 11 – PRACTICE PROBLEMS

- 1) The data below shows the favorite sport for a sample of 20 college freshmen.

Football	Soccer	Baseball	Cheerleading	Basketball	Swimming
5	4	4	2	4	1

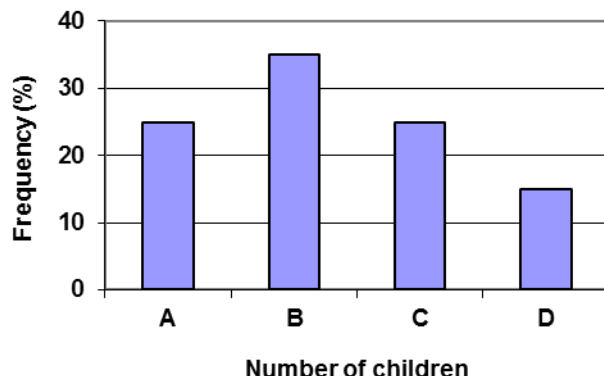
- Given the frequencies above, draw a relative frequency table for the data.
 - Construct a bar graph of the data.
 - Construct a Pareto Chart of the data.
 - Draw a pie chart of the data.
- 2) The data below shows the type of pet(s) owned by a sample of individuals.

Dog Dog Dog Cat Bird Cat Turtle Dog Cat Cat Bird Hamster
 Fish Bird Dog Dog Cat Turtle Hamster Dog Cat Dog Dog Bird

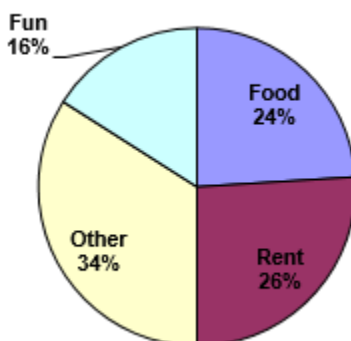
- Construct a relative frequency table for the type of pet.
 - Construct a bar graph of the data.
 - Construct a pie chart of the data.
- 3) The number of clear days in a year for several U.S. cities are provided in the table below. Construct a bar graph of the data.

City	Clear Days
Phoenix, Arizona	211
Denver, Colorado	115
Nashville, Tennessee	102
Tampa, Florida	101
Washington, DC	96
Chicago, Illinois	84
Cleveland, Ohio	66
Rochester, New York	61
Seattle, Washington	58

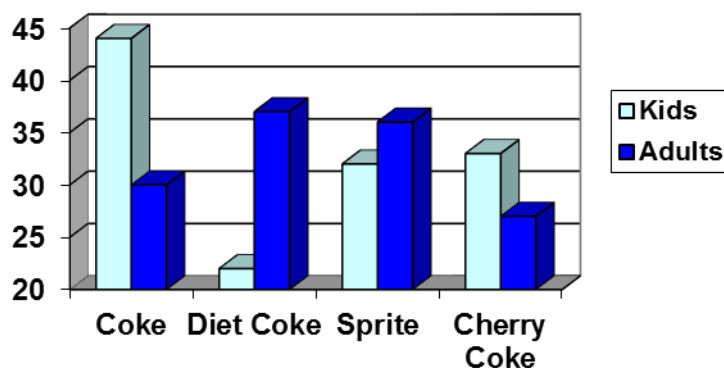
- 4) The bar graph below shows the percentage of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?



- 5) Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percentages she spent in each category are pictured here. If she spent a total of \$2600 this month, how much did she spend on rent?



- 6) A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved.



7) The table below shows scores on a Math test.

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

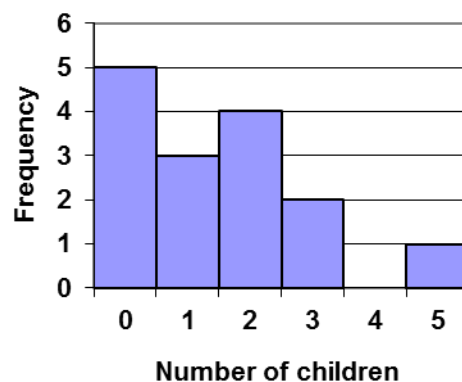
- Create a frequency table for the Math test scores.
- Construct a histogram of the data.

8) A group of adults were asked how many cars they had in their household.

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

- Create a relative frequency table for the car number data.
- Construct a histogram of the data.

9) A group of adults were asked how many children they have in their families. The graph below shows the number of adults who indicated each number of children



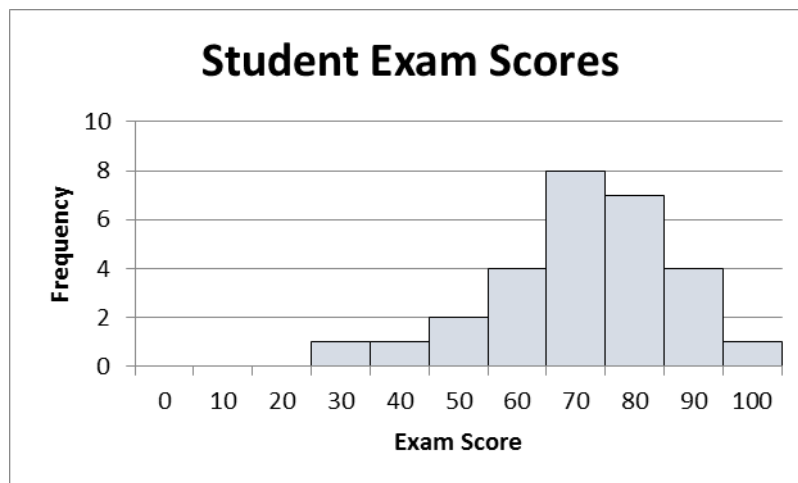
- How many adults were questioned?
- What percentage of the adults questioned had 0 children?

10) Jasmine was interested in how many days it would take an order from Amazon to arrive at her door. The graph below shows the data she collected.



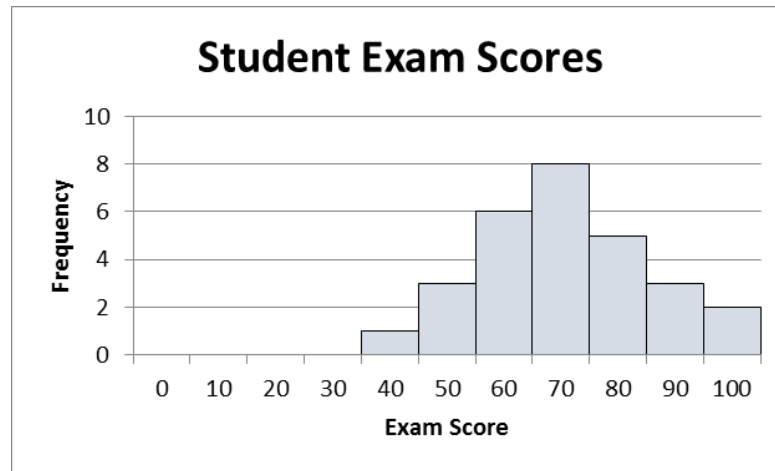
- a) How many orders did she make?
- b) What percentage of the orders arrived in one day?

11) The histogram below shows the “Exam Scores” for a sample of 28 students.



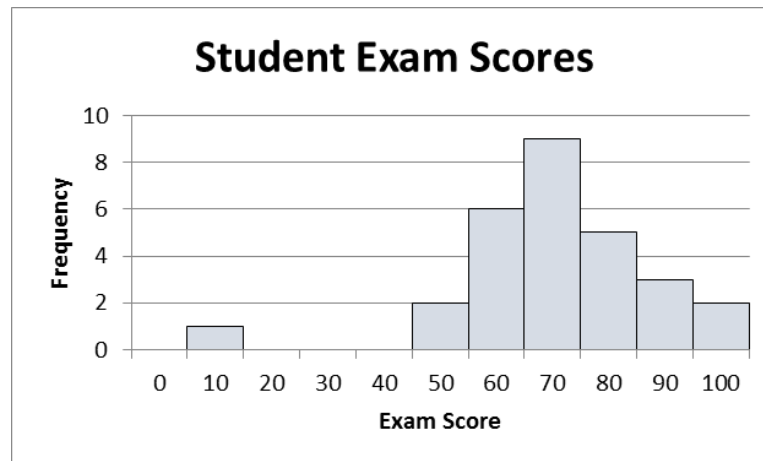
- a) Is the distribution symmetric or skewed?
- b) Classify the shape of the distribution as bell shaped, uniform, right skewed, or left skewed.
- c) Do there appear to be any outliers in the data set? Why or why not?

12) The histogram below shows the “Exam Scores” for a sample of 28 students.



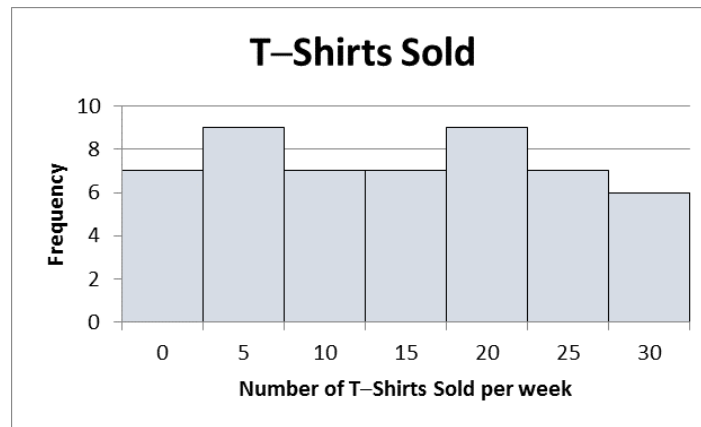
- Is the distribution symmetric or skewed?
- Classify the shape of the distribution as bell shaped, uniform, right skewed, or left skewed.
- Do there appear to be any outliers in the data set? Why or why not?

13) The histogram below shows the “Exam Scores” for a sample of 28 students.



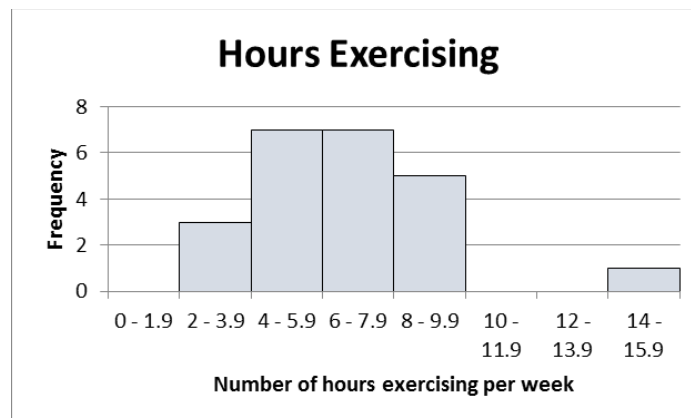
- Is the distribution symmetric or skewed?
- Classify the shape of the distribution as bell shaped, uniform, right skewed, or left skewed.
- Do there appear to be any outliers in the data set? Why or why not?

14) A student started an online T-shirt business. The histogram below shows the “Number of T-shirts Sold” over a period of 52 weeks.



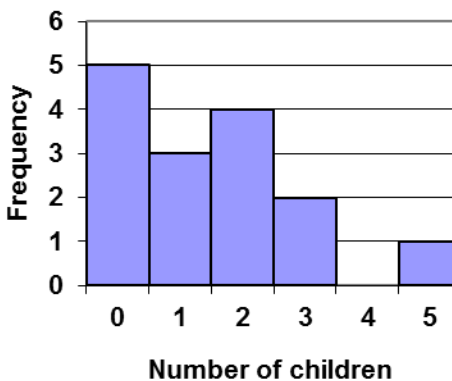
- a) Is the distribution symmetric or skewed?
- b) Classify the shape of the distribution as bell shaped, uniform, right skewed, or left skewed.
- c) Do there appear to be any outliers in the data set? Why or why not?

15) A survey was administered at a gym asking people how many hours they spend exercising in one week. The histogram below shows the “Number of Hours Exercising” for a sample of 15 people.



- a) Is the distribution symmetric or skewed?
- b) Classify the shape of the distribution as bell shaped, uniform, right skewed, or left skewed.
- c) Do there appear to be any outliers in the data set? Why or why not?

- 16) A group of diners were asked how much they would pay for a meal. Their responses were: \$7.50, \$8.25, \$9.00, \$8.00, \$7.25, \$7.50, \$8.00, \$7.00.
- Find the mean.
 - Find the median.
 - Find the standard deviation.
- 17) You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
- Find the mean.
 - Find the median.
 - Find the standard deviation.
- 18) Five real estate exam scores are 430, 430, 480, 480, and 495. Find the mode(s).
- 19) Refer back to the type of pet(s) owned by a sample of individuals:
- | | | | | | | | | | | | |
|------|------|-----|-----|------|--------|---------|-----|-----|-----|------|---------|
| Dog | Dog | Dog | Cat | Bird | Cat | Turtle | Dog | Cat | Cat | Bird | Hamster |
| Fish | Bird | Dog | Dog | Cat | Turtle | Hamster | Dog | Cat | Dog | Dog | Bird |
- What is the mode for type of pet(s) owned?
 - Why is the mode the best measure of center for this data set?
- 20) Refer to the histogram below:



- What is the mode of the number of children for the group surveyed?
- Compute the mean number of children for the group surveyed.
- Compute the median number of children for the group surveyed.

21) Refer to the histogram below:



- What is the mode of the number of shipping days?
- Compute the mean number of shipping days.
- Compute the median number of shipping days.

22) Suppose that in a small town of 50 people, one person earns \$5,000,000 per year and the other 49 each earn \$30,000. Calculate both the mean and the median. Which is the better measure of the "center": the mean or the median?

23) The following table provides the tuition rates (in \$ per credit hour) at a sample of 15 colleges and universities.

175	416	87	203	180
257	182	111	315	504
163	218	744	221	1300

- Find the mean tuition rate.
- Find the standard deviation of the tuition rate.

24) A survey of enrollment at 35 community colleges across the United States yielded the following figures:

6414; 1550; 2109; 9350; 21828; 4300; 5944; 5722; 2825; 2044; 5481; 5200;
 5853; 2750; 10012; 6357; 27000; 9414; 7681; 3200; 17500; 9200; 7380; 18314;
 6557; 13713; 17768; 7493; 2771; 2861; 1263; 7285; 28165; 5080; 11622

- Organize the data into a chart with five intervals of equal width. Label the two columns "Enrollment" and "Frequency."
- Construct a histogram of the data.
- Calculate the sample mean.
- Calculate the sample standard deviation.

25) On a baseball team, the ages of each of the players are as follows:

21; 21; 22; 23; 24; 24; 25; 25;
 28; 29; 29; 31; 32; 33; 33; 34;
 35; 36; 36; 36; 36; 38; 38; 38; 40

- Organize the data into a chart with five intervals of equal width. Label the two columns "Age" and "Frequency."
- Construct a histogram of the data.
- Calculate the sample mean.
- Calculate the sample standard deviation.

26) Twenty – five randomly selected students were asked the number of movies they watched the previous week. The results are as follows:

# of movies	Frequency
0	5
1	9
2	6
3	4
4	1

- Find the sample mean.
- Find the standard deviation.
- Find the 5 Number Summary.
- Draw the boxplot.

27) Forty randomly selected students were asked the number of pairs of sneakers they owned. Let X = the number of pairs of sneakers owned. The results are as follows:

# of pairs	Frequency
1	2
2	5
3	8
4	12
5	12
6	0
7	1

- Find the 5 Number Summary.
- Draw the boxplot.

- 28) The following table provides the tuition rates (in \$ per credit hour) at a sample of 15 colleges and universities.

175	416	87	203	180
257	182	111	315	504
163	218	744	221	1300

- a) Find the 5 Number Summary.
 - b) Draw the boxplot.
 - c) Find the 25th percentile for the data set.
 - d) Find the 50th percentile for the data set.
- 29) Following are the published weights (in pounds) of all of the team members of the San Francisco 49ers from a previous year.

177; 205; 210; 210; 232; 205; 185; 185; 178; 210; 206; 212; 184; 174; 185; 242;
188; 212; 215; 247; 241; 223; 220; 260; 245; 259; 278; 270; 280; 295; 275; 285;
290; 272; 273; 280; 285; 286; 200; 215; 185; 230; 250; 241; 190; 260; 250; 302;
265; 290; 276; 228; 265

- a) Find the mean of the data.
 - b) Find the standard deviation of the data.
 - c) Find the 5 Number Summary
 - d) Construct a box plot of the data.
- 30) Sixty – five randomly selected car salespersons were asked the number of cars they generally sell in one week. Fourteen people answered that they generally sell three cars; nineteen generally sell four cars; twelve generally sell five cars; nine generally sell six cars; eleven generally sell seven cars.
- a) Find the 5 Number Summary for this data set.
 - b) Construct a box plot for this data set.
 - c) Find the 50th percentile for the data set.
 - d) Find the 75th percentile for the data set.

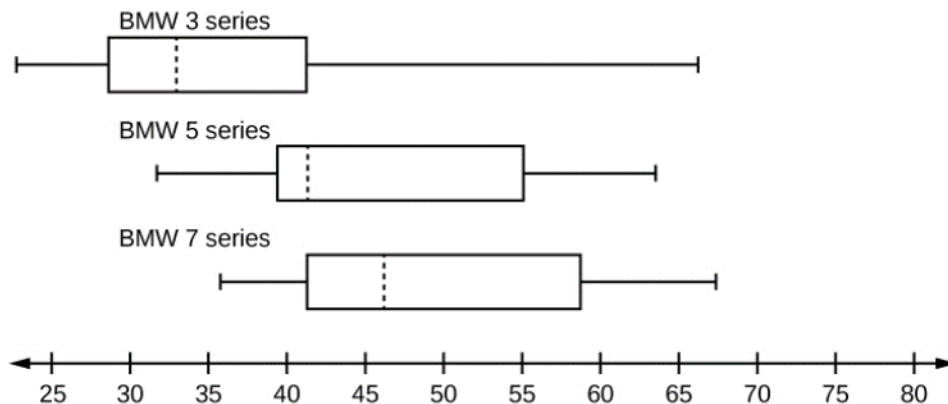
31) The box plot below shows salaries for Actuaries and CPAs.



- What is the 5 Number Summary for CPA's?
- What is the 5 Number Summary for Actuaries?
- Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?

32) Referring to the boxplot above, what percentage of actuaries make more than the median salary of a CPA?

33) A survey was conducted of 130 purchasers of new BMW 3 series cars, 130 purchasers of new BMW 5 series cars, and 130 purchasers of new BMW 7 series cars. In it, people were asked the age they were when they purchased their car. The following box plots display the results.



- Which BMW series has the largest median?
- Which BMW series has the largest standard deviation?
- As the BMW series increases from 3 to 5 to 7, the price also increases. Does that surprise you, based on the data in the boxplots? Why or why not? Explain your reasoning.

34) Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once they have started the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.

Researcher 1: 3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher 2: 3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

- a) Create comparative histograms of the data.
- b) Create comparative boxplots of the data.

CHAPTER 12 – THE NORMAL DISTRIBUTION

Chapter 12 Learning Objectives:

- 1) Be familiar with properties of the normal distribution
- 2) Be able to apply and interpret the Empirical Rule to a normally distributed variable
- 3) Calculate and interpret the meaning of z-scores
- 4) Find the area under specified regions of the standard normal distribution
- 5) Use z-scores to find the percentage of observations with a specific location for a normally distributed variable

Section 12.1 – The Normal Distribution

Section 12.2 – The Empirical Rule

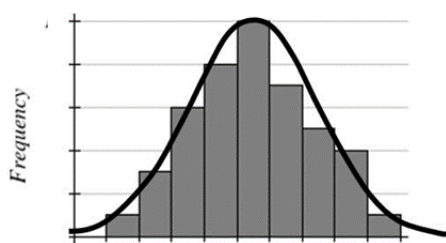
Section 12.3 – Standard Scores (z – scores)

Section 12.4 – Finding Percentages with Normally Distributed Variables

SECTION 12.1: THE NORMAL DISTRIBUTION

In chapter 10, we learned about sampling, and different ways to collect a random sample that is representative of the desired population. Although the characteristics of the population (population parameters) may be unknown, random sampling can be used to yield reliable estimates of these characteristics. Sample data can be plotted on graphs (like histograms or boxplots) to provide a visual representation of the **distribution** of data values.

Take, for example, the heights of all of the women at SCC. If we were to take a random sample of female SCC students and record their heights, we would most likely find the majority of the heights would be close to a central value, the mean, with larger or smaller values becoming less and less common. The resulting distribution might look something like this:

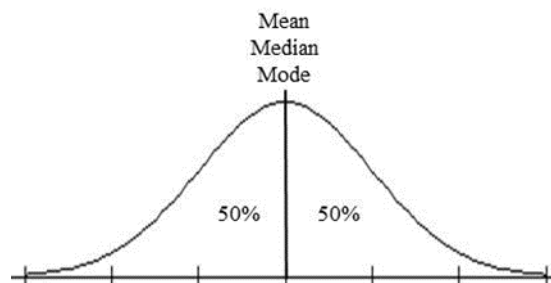


This type of symmetric, bell – shaped distribution is more formally called the **Normal Distribution**. Because actual data are almost never **exactly** normally distributed, we say that the data are **approximately** normal.

The Normal Distribution

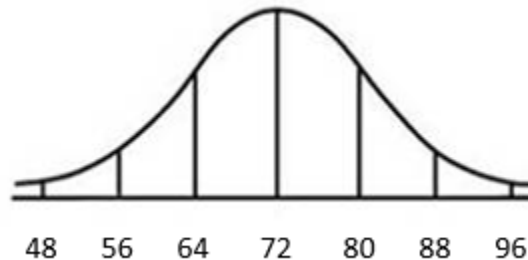
Here are a few important characteristics of the Normal Distribution:

- The distribution is symmetric about the mean, with a single peak
- The mean = median = mode of the distribution
- Because the distribution is symmetric, 50% of the data values are below the mean, and 50% of the data values are above the mean (for a total of 100% under the entire distribution curve)
- Data values farther from the mean become increasingly rare.
- The graph of the Normal Distribution is bell – shaped, with tapering tails that approach, but never actually touch the horizontal axis.
- Almost all of the area under a Normal Distribution curve is within 3 standard deviations of the mean.
- Each tick mark on the horizontal axis is marking off increments of 1 standard deviation above or below the mean.



Example 1: The test scores on a math exam are approximately normally distributed with mean 72 and standard deviation 8. Draw the associated normal distribution curve, and label the axis appropriately.

Solution: Since the test scores are approximately normally distributed, we know that the curve is bell — shaped, and centered at its mean of 72. Since the standard deviation is 8, each line on the horizontal axis is 8 units away from the other lines.

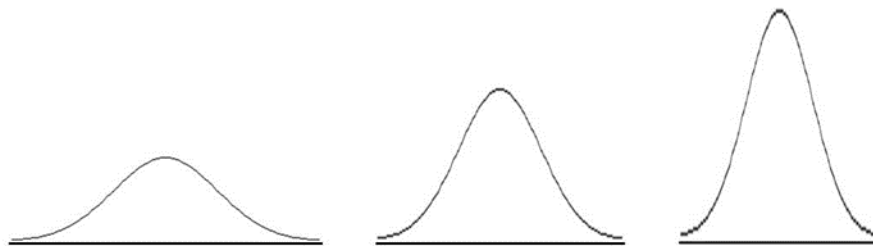


Note: When drawing a normal distribution curve, we usually extend the curve out to three standard deviations below the mean, and three standard deviations above the mean.

Section 12.1 You Try: Draw the normal distribution curve for the variable X that is approximately normally distributed with mean 15 and standard deviation 2.

SECTION 12.2: THE EMPIRICAL RULE

Consider the distributions shown below:



All of them are normally distributed, but each has different variation (spread).

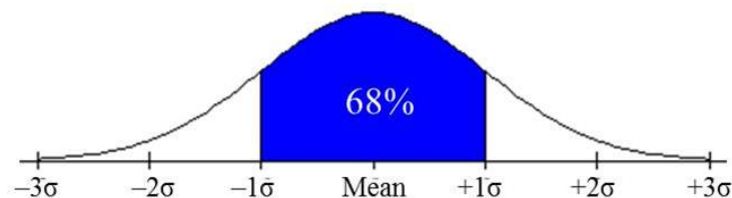
For a Normal Distribution, we use the **standard deviation** to describe the variation, or spread of the data values around the mean. The standard deviation is commonly represented by the lowercase Greek letter sigma, σ .

While each of the above distributions has a different variation, the proportion (or percentage) of the variation is still distributed in the same way for every normal distribution. That may be a bit confusing, but think about the way percentages work. Perhaps you are leaving a tip for a dinner bill, and you always leave a 20% tip. The dollar amount of the tip will be different for a \$100 dinner bill than it would be for a \$50 dinner bill, but in either case you would still leave 20% for the tip.

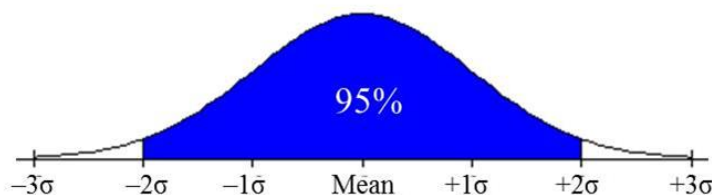
In the three normal distributions above, the variation is different for each of the 3 curves, but the percentage of the variation that is spread throughout the curve is still organized in the same way. The way the variation is distributed in the curve follows what we call the Empirical Rule (also called the 68 – 95 – 99.7 Rule).

The Empirical Rule (68 – 95 – 99.7 Rule)

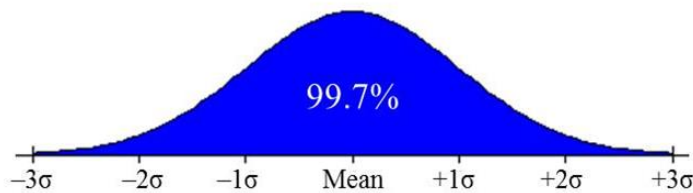
In a normal distribution:



About **68%** of the data values will fall within **1 standard deviation** of the mean.



About **95%** of the data values will be within **2 standard deviations** of the mean.



About **99.7%** of the data values will fall within **3 standard deviations** of the mean.

This rule is called the **Empirical Rule** or sometimes the 68 – 95 – 99.7 Rule

Example 2: Scores on a standardized test were normally distributed with a mean of 510 and a standard deviation of 95. Use the Empirical Rule to complete following statements.

- a) 68% of the students taking this exam scored between ____ and ____.
- b) 95% of the students taking this exam scored between ____ and ____.
- c) 99.7% of the students taking this exam scored between ____ and ____.

Solution: You will need to make some calculations. Find the mean: 510.

$$\text{Mean} + 1 \text{ standard deviation} = 510 + 95 = 605$$

$$\text{Mean} + 2 \text{ standard deviation} = 510 + 2(95) = 700$$

$$\text{Mean} + 3 \text{ standard deviation} = 510 + 3(95) = 795$$

$$\text{Mean} - 1 \text{ standard deviation} = 510 - 95 = 415$$

$$\text{Mean} - 2 \text{ standard deviation} = 510 - 2(95) = 320$$

$$\text{Mean} - 3 \text{ standard deviation} = 510 - 3(95) = 225$$

Applying the Empirical Rule, we know that:

- a) 68% of the scores will be within one standard deviation of the mean, so 68% of the students taking this exam scored between 415 and 605.
- b) 95% of the scores will be within two standard deviations of the mean, so 95% of the students taking this exam scored between 320 and 700.
- c) 99.7% of the scores will be within three standard deviations of the mean, so 99.7% of the students taking this exam scored between 225 and 795.

Section 12.2 You Try: Gear circumferences for a manufactured bicycle part were normally distributed with a mean of 34 inches and a standard deviation of 0.04 inches.

- a) Draw and label a sketch of this distribution.
- b) Use the Empirical Rule to complete following statements.

68% of the students taking this exam scored between ____ and ____.

95% of the students taking this exam scored between ____ and ____.

99.7% of the students taking this exam scored between ____ and ____.

SECTION 12.3: STANDARD SCORES (Z – SCORES)

The Empirical Rule only applies when a value is **exactly** 1, 2, or 3 standard deviations away from the mean. This is not usually the case.

A **standard score** (also called “**z-score**”) is the number of standard deviations a data value is from the mean of the distribution. We can plot z-scores on a special normal distribution called the **standard normal distribution**. The standard normal distribution is a normal distribution that always has population mean 0 and population standard deviation of 1.

Standard Scores (z-scores)

A **standard score** (also called “**z-score**”) is the number of standard deviations a specific data value is from the mean of the distribution. Standard scores (often abbreviated by the letter z) have the following characteristics:

- 1) $z = 0$ corresponds to the mean of the distribution (because the mean is 0 standard deviations from itself)
- 2) If z is positive, then the data value is above the mean.
- 3) If z is negative, then the data value is below the mean.

To calculate the z - score for any data value, x , use the following formula:

$$z = \frac{\text{data value} - \text{mean}}{\text{standard deviation}} \quad \text{or} \quad z = \frac{x - \text{mean}}{\text{standard deviation}}$$

Solving the above formula for the data value by multiplying the standard deviation and adding the mean gives the following formula for a data value given a z-score and the mean and standard deviation.

$$\text{data value} = z \times \text{standard deviation} + \text{mean}$$

or

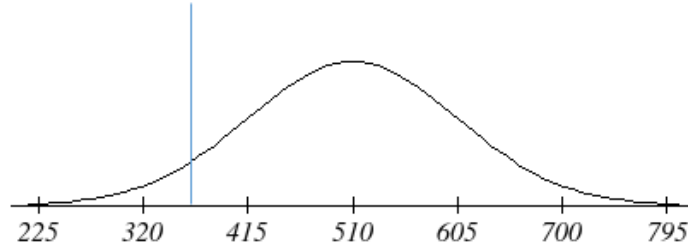
$$x = z \times \text{standard deviation} + \text{mean}$$

Applying the Empirical Rule to the standard normal distribution, we know that 68% of all z-scores will be between -1 and 1 , 95% of all z-scores will be between -2 and 2 , and 99.7% of all z-scores will be between -3 and 3 . **A z-score below -3 or above 3 is possible, but is very unlikely.**

Example 3: Scores on a standardized test were normally distributed with a mean of 510 and a standard deviation of 95.

- A student scores 365 points on the test. What is his standard score?
- What test score corresponds to a standard score of 2.2?

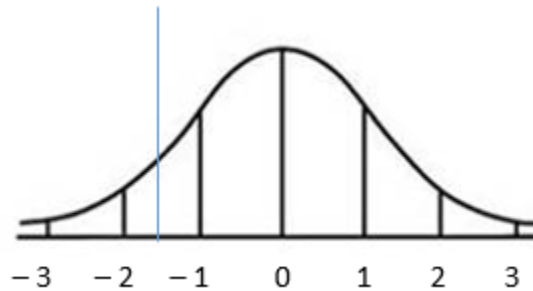
Solution: a) Here, $x = 365$, mean = 510, and standard deviation = 95.



So we have:

$$z = \frac{365 - 510}{95} = \frac{-145}{95} \approx -1.53$$

So this student's score was about 1.53 standard deviations below the mean. You can see visually that the test score of 365 corresponds exactly to the z-score of -1.53 on the standard normal distribution curve below.

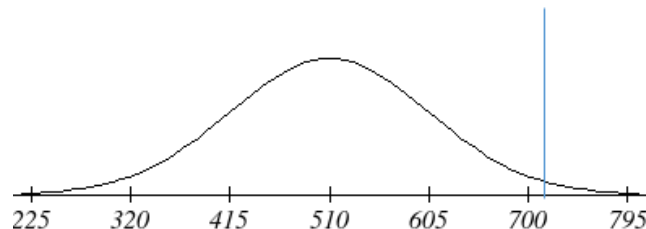


b) Here, $z = 2.2$, mean = 510, and standard deviation = 95. So

$$x = z \times \text{standard deviation} + \text{mean}$$

$$x = 2.2 \times 95 + 510$$

$$x = 209 + 510 = 719$$



So a test score of 719 corresponds to a standard score of 2.2.

Example 4: The weights of a certain dog breed are approximately normally distributed with a mean of 62 pounds, and a standard deviation of 6 pounds.

- a) A dog of this breed weighs 55 pounds. What is the z-score of the dog's weight? Is this dog's weight unusual for this breed?
- b) A dog's weight has a z-score of -3.2 . What is the dog's weight?

Solution: a) Here, $x = 55$, mean = 62, and standard deviation = 6. So we have:

$$z = \frac{55 - 62}{6} = \frac{-7}{6} \approx -1.17$$

So this dog's weight was about 1.17 standard deviations below the mean. No, the dog's weight is well within 3 standard deviations of the mean, so it is not unusual.

b) Here, $z = -3.2$, mean = 62, and standard deviation = 6. We want the dog's weight. So

$$x = z \times \text{standard deviation} + \text{mean}$$

$$x = -3.2 \times 6 + 62$$

$$x = -19.2 + 62 = 42.8$$

So a z-score of -3.2 corresponds to a dog that weighs 42.8 pounds. Since this z-score is less than -3 , this is an unusually low z-score. This indicates that this dog's weight is unusually low.

Section 12.3 You Try: The lengths of mature trout in a local lake are approximately normally distributed with a mean of 13.2 inches, and a standard deviation of 1.8 inches.

- Draw and label the distribution of trout length.
- Draw and label the standard normal distribution.
- Find the z-score corresponding to a fish that is 19 inches long.
- How long is a fish that has a z-score of -0.85 ?
- Draw the z-score from part c on your graph from part b.

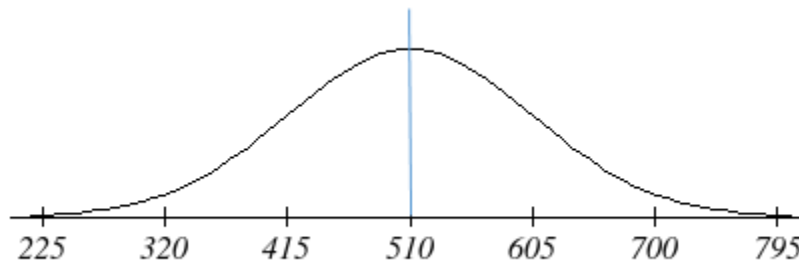
Now we know how to convert any value of a variable that is approximately normally distributed into a z-score. This z-score represents a location on the standard normal distribution. You may be thinking, ok, so I can convert an observed value of a variable into a z-score, but why is that useful? In the next section we will begin to answer this question.

SECTION 12.4: FINDING PERCENTAGES WITH THE NORMAL DISTRIBUTION

We will start out this section with a question: Why go to the trouble of calculating the z-score that corresponds to an observed value? We will answer this question through the use of the following examples:

Example 5: Consider the data set consisting of scores on a standardized test. The test scores were normally distributed with a mean of 510 and a standard deviation of 95. What percentage of students scored less than 510 on the test?

Solution: We can answer this question using what we already know about normal distributions. Specifically, they are centered about their mean, and symmetric about their mean. If we draw a line to divide the data set at the mean, it is clear that the percentage of observations that are less than the mean should be 50% (with another 50% being greater than the mean).



Example 6 Part 1: Consider again the data set consisting of scores on a standardized test were normally distributed with a mean of 510 and a standard deviation of 95. What percentage of students scored less than 650 on the test?

Solution: This is the same question we just answered, but you can see quickly that changing the observed value from 510 to 650 made this question much more difficult to answer. We don't know the exact percentage of scores that are less than 650.

If we had a list of all of the test scores, we could count how many of them were below 650. We could then divide by the total number of scores, and convert to a percentage. That sounds simple enough, but we don't have the list of test scores. So how can we answer this question knowing just the shape of the distribution, the mean, and the standard deviation?

We will come back to this example and answer this question.

At this point, in order to continue solving the problem above, we have to know how to find the percentage of values (areas) under the normal distribution curve with mean 510 and standard deviation 95. That involves complex formulas and calculus. To avoid those requirements, we will use the standard normal distribution curve. Because we learned in the last section how to convert any normally distributed variable into the standard normal, all we need to know are percentages (or areas) under the standard normal distribution curve. These values can be readily looked up using a z-table, or can be found using your graphing calculator.

The following z-table provides areas (percentages of values) under the standard normal curve that lie to the left of specified z-scores.

Abbreviated Table of Areas under the Standard Normal Curve

Z – Score	Area	Z – Score	Area	Z – Score	Area	Z – Score	Area
–3.0	0.0013	–1.5	0.0668	0.0	0.5000	1.5	0.9332
–2.9	0.0019	–1.4	0.0808	0.1	0.5398	1.6	0.9452
–2.8	0.0026	–1.3	0.0968	0.2	0.5793	1.7	0.9554
–2.7	0.0035	–1.2	0.1151	0.3	0.6179	1.8	0.9641
–2.6	0.0047	–1.1	0.1357	0.4	0.6554	1.9	0.9713
–2.5	0.0062	–1.0	0.1587	0.5	0.6915	2.0	0.9773
–2.4	0.0082	–0.9	0.1841	0.6	0.7257	2.1	0.9821
–2.3	0.0107	–0.8	0.2119	0.7	0.7580	2.2	0.9861
–2.2	0.0139	–0.7	0.2420	0.8	0.7881	2.3	0.9893
–2.1	0.0179	–0.6	0.2743	0.9	0.8159	2.4	0.9918
–2.0	0.0228	–0.5	0.3085	1.0	0.8413	2.5	0.9938
–1.9	0.0287	–0.4	0.3446	1.1	0.8643	2.6	0.9953
–1.8	0.0359	–0.3	0.3821	1.2	0.8849	2.7	0.9965
–1.7	0.0446	–0.2	0.4207	1.3	0.9032	2.8	0.9974
–1.6	0.0548	–0.1	0.4602	1.4	0.9192	2.9	0.9981

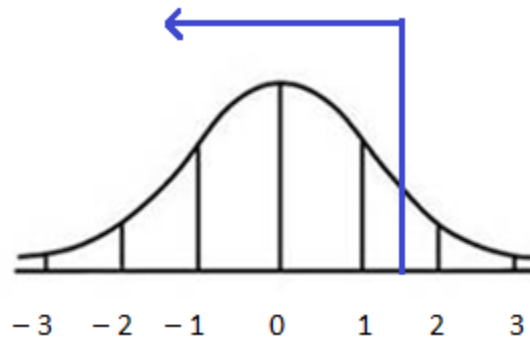
Note: All values are the area to the LEFT of the specified z-score. Area to the Right = 1 – Area to the Left

Example 6 Part 2: Let's go back to the previous example, using what we now know about finding percentages under the standard normal distribution curve. Our data consisted of scores on a standardized test that were normally distributed with a mean of 510 and a standard deviation of 95. We wanted to know what percentage of students scored less than 650 on the test.

Solution: As starting point, we will calculate the z-score that corresponds to a test score of 650.

$$z = \frac{650 - 510}{95} = \frac{140}{95} \approx 1.47$$

This tells us that the percentage of students who score less than 650 on the test will be equal to the percentage of observations that are less than 1.47 under the standard normal curve.



Using the z- table: Look in the z-score columns to find the integer and first decimal in your z-score. For this example, we will round 1.47 to 1.5. The area from the table is .9332. We can say that approximately 93.32% of all test scores were less than 650.

Using your graphing calculator: Press the “2nd” key, followed by the “Distr” key (below and to the left of the arrows). Choose option “2:normalcdf(” and press “Enter”. The calculator will ask you for the lower bound. This should be the z – score that is the leftmost boundary of the area you are trying to find. Since in this case we want all the area to the left of 1.47, there is no lower bound. Type $-1,000,000$ as your lower value. The calculator will ask you for the upper bound. In this example, the upper bound is 1.47 since we want the percentage of all values less than 1.47. You should enter 0 for the mean, and 1 for the standard deviation since this is a standard normal distribution. Press “Enter”, when you see a blinking cursor press “Enter” again. Your calculator should output 0.9292190874. We would round this to 0.9292, or 92.92%. The table on the next page will walk you through this process, in general.

NOTE: The table will give slightly different answers than the calculator. The table is an approximation (with rounding error) to the actual percentages, while the calculator gives you a much more accurate answer.

Finding Areas under the Normal Curve on Your Graphing Calculator

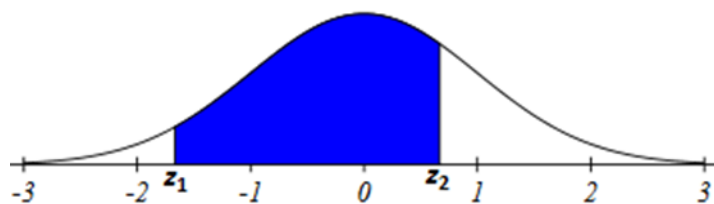
Press the “2nd” key, followed by the “Distr” key (below and to the left of the arrows). Choose option “2:normalcdf(” and press “Enter”. The calculator will ask you for the lower bound, the upper bound, the mean, and the standard deviation.

normalcdf(lower bound, upper bound, mean, standard deviation)

Since we are working with the *Standard Normal Distribution*, the mean = 0 and the standard deviation = 1.

Find the shaded areas below.

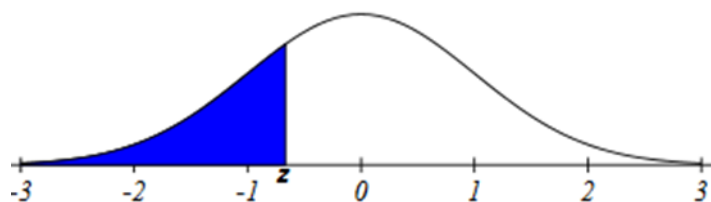
- 1) Find the area between two z-scores, z_1 and z_2



Lower bound = z_1
 Upper bound = z_2
 Mean = 0
 Standard Deviation = 1
normalcdf($z_1, z_2, 0, 1$)

- 2) Find the area below z .

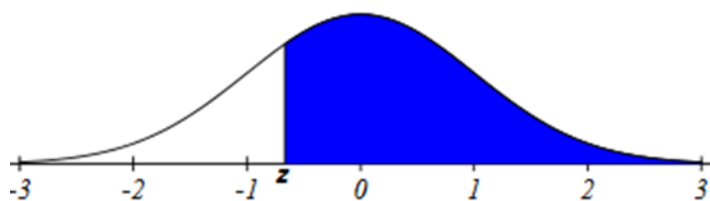
NOTE: Since there is no lower bound, we choose $-1,000,000$ for the lower bound.



Lower bound = $-1,000,000$
 Upper bound = z
 Mean = 0
 Standard Deviation = 1
normalcdf($-1000000, z, 0, 1$)

- 3) Find the area above z .

NOTE: Since there is no upper bound, we choose $1,000,000$ for the upper bound.



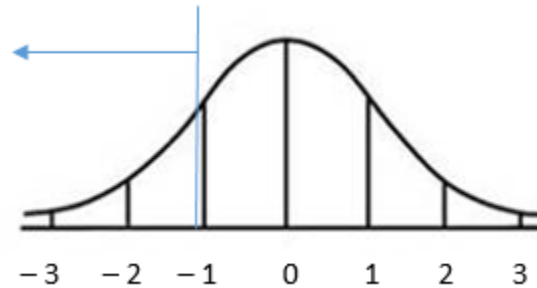
Lower bound = z
 Upper bound = $1,000,000$
 Mean = 0
 Standard Deviation = 1
normalcdf($z, 1000000, 0, 1$)

As a shortcut, you can also use the “normalcdf” option with the original values of X , and the mean and standard deviation of the given normally distributed variable (without first converting to z-scores). In the following examples, we will show both methods.

Example 7: The weights of a certain dog breed are approximately normally distributed with a mean of 62 pounds, and a standard deviation of 6 pounds. Find the percentage of dogs of this breed that weigh less than 55 pounds.

Solution: We need to start out by finding the z-score that corresponds to a dog weighing 55 pounds.

$$z = \frac{55 - 62}{6} = \frac{-7}{6} = -1.17$$



Using the Table: In the abridged z-table, we don't have -1.17. The closest value to -1.17 is -1.2. Our answer will be an approximation. The corresponding area to the left is 0.1151. The percentage of dogs of this breed that weigh less than 55 pounds is approximately 11.51%.

Converting to z-scores and using the Calculator: In the graphing calculator, under 2nd Distr we select 2:normalcdf. We then need to enter -1,000,000 again for the lower bound. For the upper bound we will enter -1.17. Since we converted the dog weight into a z-score, we will use a mean of 0, and a standard deviation of 1. The resulting output is 0.1210. The percentage of dogs of this breed that weigh less than 55 pounds is about 12.10%.

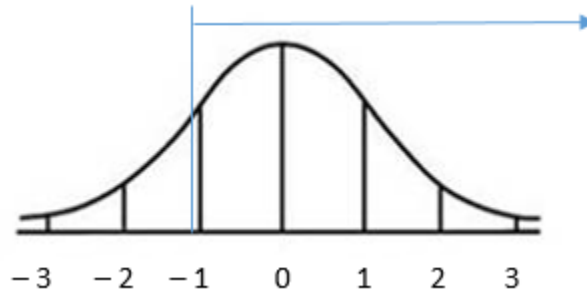
Using the Calculator without Converting to z-scores: As an alternative method, once we understand the process of finding area under the standard normal distribution, we can use the original value of dog weight, along with the original mean dog weight and original standard deviation of dog weights to find the area under the normal distribution curve that lies below 55. This will save you lots of time! In the graphing calculator, under 2nd Distr we select 2:normalcdf. We then need to enter -1,000,000 again for the lower bound. For the upper bound we will enter 55. Since we are using dog weights now, not z-scores, we will use a mean dog weight of 62 pounds, and the standard deviation of dog weights of 6 pounds. The resulting output is 0.1217 (a slightly different estimate than the z-score method). The percentage of dogs of this breed that weigh less than 55 pounds is about 12.17%.

Example 8: The weights of a certain dog breed are approximately normally distributed with a mean of 62 pounds, and a standard deviation of 6 pounds. Find the percentage of dogs of this breed that weigh MORE than 55 pounds.

Solution: We need to start out by finding the z-score that corresponds to a dog weighing 55 pounds.

$$z = \frac{55 - 62}{6} = \frac{-7}{6} = -1.17$$

This time we need to find the percentage of z-scores that are greater than -1.17 (or to the right of -1.17).



Converting to z-scores and using the Calculator: In the graphing calculator, under 2nd Distr, 2:normalcdf, we need to enter -1.17 for the lower bound. For the upper bound we will enter $1,000,000$. The resulting output is 0.8790 . The percentage of dogs of this breed that weigh MORE than 55 pounds is 87.90% .

Recall, the total area under a distribution curve is 1 (or 100%). In Example 7 we found the area to the LEFT of $z = -1.17$ and in Example 8 we found the area to the RIGHT of $z = 1.17$. Notice that the area to the left (0.1210) plus the area to the right (0.8790) of the same z-score will always sum to 1.

Using the Calculator without Converting to z-scores: In the graphing calculator, under 2nd Distr, 2:normalcdf, we need to enter 55 for the lower bound (since we are using dog weights, not z-scores). For the upper bound we will enter $1,000,000$. The resulting output is 0.8783 . The percentage of dogs of this breed that weigh MORE than 55 pounds is 87.83% . Again, this method gives you slightly different estimate of the area.

The previous example illustrates an important property about areas under the standard normal distribution. Since the total percentage of values under the distribution curve is always equal to 100% (or 1) the area to the left of a z-score plus the area to the right of that same z-score will always equal 1 (or 100% of the observations).

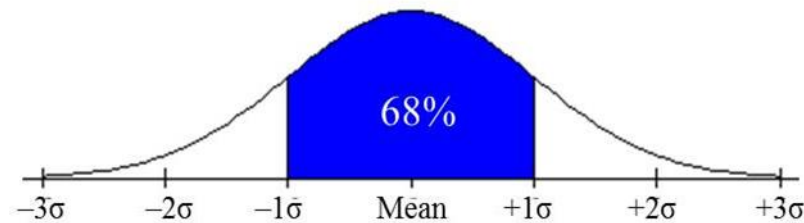
Section 12.4A You Try: The annual rainfall in a certain region is approximately normally distributed with mean 33.5 inches and standard deviation 5.6 inches.

- What percentage of years will have an annual rainfall of less than 30 inches?
- What percentage of years will have an annual rainfall of more than 40 inches?

Finding the Area between Two Z-Scores

Recall that in Section 12.2 we learned the Empirical Rule (or 68 – 95 - 99.7 Rule). This rule allowed us to find the percentage of observations that lied between specific values of our variable. While this is useful, there are many occasions where the data values we are interested in do not happen to fall right at 1, 2, or 3 standard deviations away from the mean.

We know from the Empirical Rule that about 68% of all observations will fall between -1σ and 1σ (between -1 and 1 in the standard normal distribution)

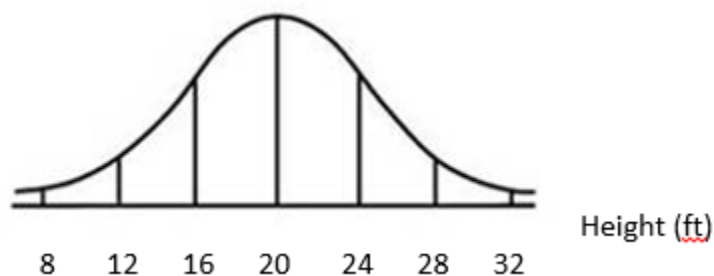


What if we wanted to find the percentage of all observations that will have z-scores between -2 and 2 ? Again, using the Empirical Rule, we know the answer is approximately 95%.

What if we wanted to find the percentage of all observations that will have z-scores between -1.5 and 1.5 ? This becomes much more challenging. We would expect the answer to be between 68% and 95%, but we can now get much more precise.

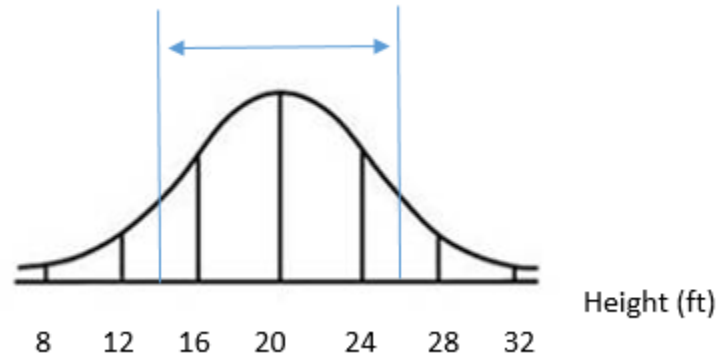
Example 9: A scientist finds that the heights of a certain tree are approximately normally distributed with mean 20 feet and standard deviation 4 feet.

The associated normal distribution curve is as follows:



Suppose we are asked to find the percentage of all such trees that are between 14 and 26 feet tall.

Solution: The percentage of trees that are between 14 and 26 feet tall corresponds to the percentage of values that are between the two lines drawn on the curve below:



We can find the associated z-scores that correspond to tree heights of 14 ft. and 26 ft. using the z-score formula from the previous section:

$$z = \frac{14 - 20}{4} = \frac{-6}{4} = -1.5 \qquad z = \frac{26 - 20}{4} = \frac{6}{4} = 1.5$$

We now know that the percentage of trees between 14 and 26 feet tall is the same as the percentage of observations between -1.5 and 1.5 under the standard normal distribution.

Using the table, the area that we are looking for is the area between -1.5 and 1.5. If we subtract the area to the left of -1.5 from the area to the left of 1.5, our answer will be just the remaining area between the two z-scores.

Area to the left of -1.5 = 0.0668

Area to the left of 1.5 = 0.9332

Area in between -1.5 and 1.5 = $0.9332 - 0.0668 = 0.8664$. Therefore, 86.64% of trees are between 14 and 26 feet tall.

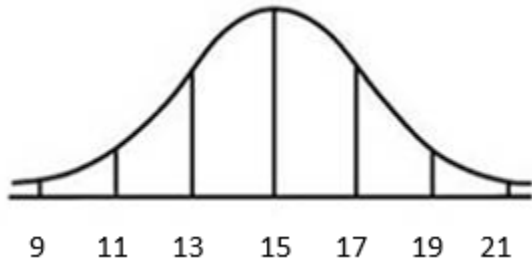
Converting to z-scores and using the Calculator: 2nd Distr, 2:normalcdf, our lower bound would be -1.5 and our upper bound would be 1.5. Since we are working with a standard normal distribution, the mean is 0 and the standard deviation is 1. The resulting percentage of trees that are between 14 and 26 feet tall is 86.64%.

Using the Calculator without Converting to z-scores: 2nd Distr, 2:normalcdf, our lower bound would be 14 and our upper bound would be 26. The mean tree height is 20 feet and the standard deviation of tree heights is 4 feet. The resulting percentage of trees that are between 14 and 26 feet tall is 86.64%.

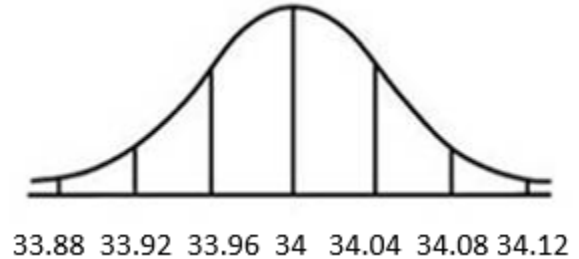
Section 12.4B You Try: The annual rainfall in a certain region is approximately normally distributed with mean 33.5 inches and standard deviation 5.6 inches. What percentage of years will have an annual rainfall of between 30 and 40 inches?

Chapter 12 – Answers to You Try Problems

12.1:

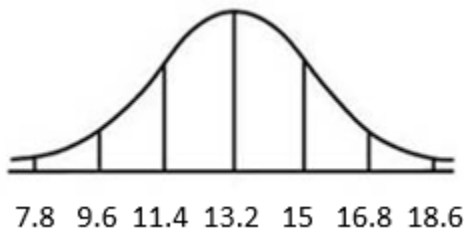


12.2a:

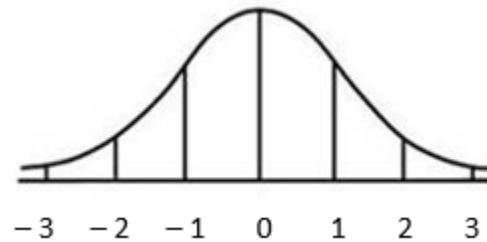


12.2b: 68% of the gears have circumferences between 33.96 and 34.04 inches
 95% of the gears have circumferences between 33.92 and 34.08 inches
 99.7% of the gears have circumferences between 33.88 and 34.12 inches

12.3a:



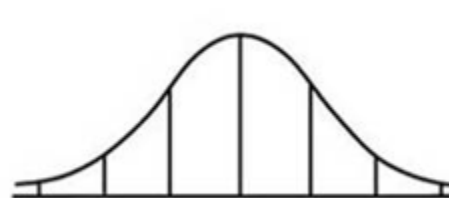
12.3b:



12.3c: $z = 3.22$

12.3d: 11.67 inches

12.3e:



12.4A. a) $z = -0.63$. Area to the left of -0.63 is 0.2643, or 26.43%

b) $z = 1.16$. Area to the right of 1.16 is 0.1230, or 12.30%

12.4B. $0.8770 - 0.2643 = .6127$. In this region, 61.27% of all years will have annual rainfall between 30 and 40 inches.

CHAPTER 12 – PRACTICE PROBLEMS

- 1) A normal distribution has a mean of 61 and a standard deviation of 15. What is the median?
- 2) In a normal distribution, what percentage of observations will lie to the left of the mean?
- 3) Suppose a normal distribution has a mean of six and a standard deviation of 1.5. Draw the associated normal distribution curve, labeling the axis appropriately.
- 4) Suppose a normal distribution has a mean of 45 and a standard deviation of 10. Draw the associated normal distribution curve, labeling the axis appropriately.
- 5) Suppose a normal distribution has a mean of -5 and a standard deviation of 2. Draw the associated normal distribution curve, labeling the axis appropriately.
- 6) Two normal distributions have the same means, but different standard deviations. Distribution A has a standard deviation of 10, and Distribution B has a standard deviation of 15. Which curve has wider spread along the horizontal axis? Why?
- 7) About what percent of x values from a normal distribution lie within one standard deviation (left and right) of the mean of that distribution?
- 8) About what percent of the x values from a normal distribution lie within two standard deviations (left and right) of the mean of that distribution?
- 9) About what percent of x values lie between the second and third standard deviations (both sides)?
- 10) Suppose the variable X is normally distributed with mean 15 and standard deviation 3. Between what x values does 68% of the data lie? Note: The range of x values is centered at the mean of the distribution.
- 11) Suppose X is normally distributed with mean -3 and standard deviation 1. Between what x values does 95% of the data lie? Note: The range of x values is centered at the mean of the distribution.
- 12) About what percent of x values lie between the mean and three standard deviations?
- 13) About what percent of x values lie between the mean and one standard deviation?
- 14) About what percent of x values lie between the first and second standard deviations from the mean (both sides)?

- 15) About what percent of x values lie between the first and third standard deviations (both sides)?
- 16) What does a z – score measure?
- 17) Consider the standard normal distribution. The mean is always ____ and the standard deviation is always ____.
- 18) What is the z – score of $x = 12$, if it is two standard deviations to the right of the mean?
- 19) What is the z – score of $x = 9$, if it is 1.5 standard deviations to the left of the mean?
- 20) What is the z – score of $x = -2$, if it is 2.78 standard deviations to the right of the mean?
- 21) What is the z – score of $x = 7$, if it is 0.133 standard deviations to the left of the mean?
- 22) Suppose X is normally distributed with mean -1 and standard deviation 2. What is the z – score of $x = 2$?
- 23) Suppose a variable is approximately normally distributed with mean 12 and standard deviation 6. What is the z – score of $x = 2$?
- 24) Suppose a variable is normally distributed with mean 9 and standard deviation 3. What is the z – score of $x = 9$?
- 25) Suppose a normal distribution has a mean of six and a standard deviation of 1.5. What is the z – score of $x = 5.5$?
- 26) In a normal distribution, $x = 5$ and $z = -1.25$. This tells you that $x = 5$ is ____ standard deviations to the ____ (right or left) of the mean.
- 27) In a normal distribution, $x = 3$ and $z = 0.67$. This tells you that $x = 3$ is ____ standard deviations to the ____ (right or left) of the mean.
- 28) In a normal distribution, $x = -2$ and $z = 6$. This tells you that $x = -2$ is ____ standard deviations to the ____ (right or left) of the mean.
- 29) In a normal distribution, $x = -5$ and $z = -3.14$. This tells you that $x = -5$ is ____ standard deviations to the ____ (right or left) of the mean.
- 30) In a normal distribution, $x = 6$ and $z = -1.7$. This tells you that $x = 6$ is ____ standard deviations to the ____ (right or left) of the mean.

College Math – Chapter 12 Practice Problems

- 31) The life of Sunshine DVD players is normally distributed with mean of 4.1 years and a standard deviation of 1.3 years. A DVD player is guaranteed for three years. We are interested in the length of time a DVD player lasts. Find the z – score corresponding to the guaranteed life of 3 years.
- 32) The heights of the 430 National Basketball Association players were listed on team rosters at the start of the 2005–2006 season. The heights of basketball players have an approximately normal distribution with mean 79 inches and a standard deviation 3.89 inches. For each of the following heights, calculate the z – score and interpret it using complete sentences.
- a) 77 inches
 - b) 85 inches
- 33) The systolic blood pressure (given in millimeters) of males has an approximately normal distribution with mean 125 and standard deviation 14. Systolic blood pressure for males follows a normal distribution.
- a) Calculate the z – scores for the male systolic blood pressures 100 and 150 millimeters.
 - b) If a male friend of yours said he thought his systolic blood pressure was 2.5 standard deviations below the mean, but that he believed his blood pressure was between 100 and 150 millimeters, what would you say to him?
- 34) A variable is normally distributed with mean 2 and standard deviation 6. What value of x has a z – score of three?
- 35) Suppose a variable has mean 8, standard deviation 1, and is normally distributed. What value of x has a z – score of -2.25 ?
- 36) Suppose a normally distributed variable has mean 4 and standard deviation 2. What value of x is 1.5 standard deviations to the left of the mean?
- 37) A variable is normally distributed with mean 4 and standard deviation 2. What value of x is two standard deviations to the right of the mean?
- 38) The life of Sunshine DVD players is normally distributed with mean of 4.1 years and a standard deviation of 1.3 years. How long does a DVD player last if its z – score is -2.0 ?

- 39) The heights of the 430 National Basketball Association players were listed on team rosters at the start of the 2005–2006 season. The heights of basketball players have an approximately normal distribution with mean 79 inches and a standard deviation 3.89 inches.
- If an NBA player reported his height had a z – score of 3.5, would you believe him? Explain your answer.
 - If an NBA player reported his height had a z – score of -3.5 , would you believe him? Explain your answer.
- 40) The systolic blood pressure (given in millimeters) of males has an approximately normal distribution with mean 125 and standard deviation 14.
- Find the systolic blood pressure of a man with a z – score of 0.75. Is this an unusual systolic blood pressure? Why or why not?
 - Find the systolic blood pressure of a man with a z – score of 3.2. Is this an unusual systolic blood pressure? Why or why not?
- 41) If the area to the left of X in a normal distribution is 0.123, what is the area to the right of x ?
- 42) If the area to the right of x in a normal distribution is 0.543, what is the area to the left of x ?
- 43) A variable is normally distributed with mean 16 and standard deviation 3. Find each of the following. Draw the associated normal curve, shading the solution region.
- Find the area to the left of 12.
 - Find the area to the left of 20.
 - Find the area to the right of 15.
 - Find the area to the right of 24.
 - Find the area between 12 and 20.
- 44) A variable is normally distributed with mean 84.6 and standard deviation 19.1. Find each of the following. Draw the associated normal curve, shading the solution region.
- Find the area to the left of 60.5.
 - Find the area to the right of 60.5.
 - Find the area to the left of 128.0.
 - Find the area to the right of 119.3.
 - Find the area between 80 and 90.

- 45) A variable is normally distributed with mean 45 and standard deviation 4.6. Find each of the following. Draw the associated normal curve, shading the solution region.
- Find the area to the left of 27.
 - Find the area to the left of 45.
 - Find the area to the right of 52.
 - Find the area to the right of 38.
 - Find the area between 33 and 48.
- 46) The patient recovery time from a particular surgical procedure is normally distributed with a mean of 5.3 days and a standard deviation of 2.1 days.
- What is the probability of spending less than two days in recovery?
 - What is the probability of spending more than two days in recovery?
 - What is the probability of spending more than 10 days in recovery?
- 47) The height of 9th grade boys at a local high school is normally distributed with an average of 66 inches and a standard deviation of 2.5 inches. Suppose one 9th grade boy is randomly chosen. Let X = height of the individual.
- Find the probability that the boy is between 65 and 69 inches. Include a sketch of the graph.
 - Would you expect to meet many 9th grade boys over 72 inches? Explain why or why not, and justify your answer numerically.
- 48) IQ is normally distributed with a mean of 100 and a standard deviation of 15. Suppose one individual is randomly chosen. Let X = IQ of an individual.
- Find the probability that the person has an IQ greater than 120. Include a sketch of the graph.
 - Find the probability that the person has an IQ less than 60. Include a sketch of the graph.
- 49) The percent of fat calories that a person in America consumes each day is normally distributed with a mean of about 36 and a standard deviation of 10. Suppose that one individual is randomly chosen. Let X = percent of fat calories.
- Find the probability that the percent of fat calories a person consumes is more than 40. Graph the situation. Shade in the area to be determined.
 - Find the probability that the percent of fat calories a person consumes is between 10 and 20. Graph the situation. Shade in the area to be determined.

- 50) Suppose that the distance of fly balls hit to the outfield (in baseball) is normally distributed with a mean of 250 feet and a standard deviation of 50 feet.
- a) If one fly ball is randomly chosen from this distribution, what is the probability that this ball traveled fewer than 220 feet? Sketch the graph. Shade the area to be determined.
 - b) If one fly ball is randomly chosen from this distribution, what is the probability that this ball traveled more than 350 feet? Sketch the graph, and shade the area to be determined.

APPENDIX A: ANSWERS TO PRACTICE PROBLEMS**Chapter 1**

- | | | | |
|---|---|--------------------------|--------------------|
| 1) 96 in | 2) 15 ft. | 3) 216 in | 4) 3 yd |
| 5) 357,456 yd | 6) 17,436,672 in | 7) 110 in | 8) 10.08 ft. |
| 9) 64 oz | 10) 2.38 lb | 11) 0.013 tons | 12) 9 lbs. 4 oz |
| 13) 12 lbs 2 oz | 14) 3.04 gal | 15) 149.84 gal | 16) 288 oz. |
| 17) 27.6 cups | 18) 3-quart saucepan (2.25 qt of soup) | | 19) 700 cm |
| 20) 3.52 L | 21) 4,830 mL | 22) 0.004 m | 23) 4.15 m |
| 24) 1.84 kg | 25) 6,000 g | 26) 3.64 cm | 27) 35,200 mg |
| 28) 22 L | 29) 9.5 km | 30) 2.96 kg | 31) 2.36 inches |
| 32) 11.43 cm | 33) 382.76 yards | 34) 11.59 km | 35) 74.84 kg |
| 36) 1.76 pounds | 37) 60.56 liters | 38) 437.45 feet | 39) 12.22 °C |
| 40) 23.89 °C | 41) 10.4 °F | 42) 98.6 °F | 43) 722.64 euros |
| 44) Rate: $\frac{540 \text{ ft}}{30 \text{ sec}}$ | Unit rate: $\frac{18 \text{ ft}}{1 \text{ sec}}$ or 18 ft. per second | 45a) 863.64 mi | b) 37.63 gal |
| 46) 938.7 ft. | 47) 290.4 ft. | 48) 66.7 m | 49) 55.6 m |
| 50) 53.33 oz. | 51) One quart | 52) \$70 | 53) \$6,000 |
| 54) 972 calories | 55) 73.3 turbines | 56) 2,666.67 dump trucks | 57) \$3,003 |
| 58) 4 lemons | 59) 64 mg | 60) 132 ft. | 61) 3,300 ft. |
| 62) 34 min | 63) 3.7 yd ³ | 64) \$5.53 per gallon | 65) \$3.62 per oz. |

Chapter 2

$$1) x = 3 \quad 2) x = -\frac{16}{7} \quad 3) k = 4 \quad 4) k = 2 \quad 5) x = -24$$

$$6) x = 18 \quad 7) x = -17.64 \quad 8) t = 25 \quad 9) t = 180$$

10) If John earns \$1000 for the week, he has sold 16 vacuums.

11) Paul needs to sell 240 bottles of water to break even.

12) A triangle with a base of 18 inches and area 84.6 square inches has a height of 9.4 inches.

13) Rate of change = 3 Initial value = 7

14) Rate of change = -10 Initial value = -9

15) Rate of change = -5 Initial value = 0

16) Rate of change = 0 Initial value = 1

17) 3 18) -2 19) 0

20) Rate of change = 21 Initial value = 14

21) Rate of change = -39 Initial value = -6

22) Rate of change = 38 Initial value = 62

23) The value of the investment decreases by \$5,400 every year.

24) The value of the investment increases by \$2,500 every year.

25) Paul's profit will increase by \$1.40 for each additional bottle of water sold.

26) a) $N = 44t + 440$ b) $N = -32t + 440$ c) $N = 440$

27) a) $P = 1700x + 59000$ b) 65,800 people c) 2014

28) a) $T = 30n + 1300$ b) \$133,300 c) 4,100 toys

29) -77.6 feet per year

30) a) \$1,400 per year b) Increasing

31) a) \$1,600 per year b) Increasing

32) a) \$13 per credit hour per year b) $C = 13n + 105$ c) \$326 d) 2014

33) 28 lunches; \$2

34a) Slim Gym (Phoenix Fitness = \$519; Slim Gym = \$490)

34b) Phoenix Fitness (Phoenix Fitness = \$939; Slim Gym = \$970)

Chapter 3

- 1) a) 75% b) 80% c) 70%
- 2) 71.4% 3) 77.8% 4) 18.2%
- 5) a) 23% b) 21.5% c) 109%
- 6) a) 0.15 b) 0.04 c) 0.075
- 7) $\frac{1}{5}$ 8) $\frac{43}{50}$ 9) $\frac{29}{50}$ 10) 28 11) 69.75 12) 46.67%
- 13) 69.39% 14) 60 15) 200 16) 30.9% 17) 94.5% 18) 38
- 19) \$5.25 20) \$20.65 21) \$75.96 22) \$243,750 23) 53.2%
- 24) Absolute change = $-\$0.12$ Percent change = -3.8%
- 25) Absolute change = $\$17,000$ Percent change = 54.8%
- 26) 20% 27) Increase = 84 parts per million Percent increase = 27.4%
- 28a) No, we only know their absolute decrease in weight. 28b) Johnny (6.5%)
- 29a) 2 percentage points 29b) 12.5%
- 30) \$4,333.70 31) \$11.70 32) \$13.91 33) \$57.50 34) \$38.00
- 35) 194 lbs. 36) \$26,000 37) \$130 38) \$17,028.27
- 39a) 64.3 b) 24 c) 12.44
- 40a) 4.63 b) 5 c) 6.30
- 41a) 170 b) Decay c) Decay factor is 0.8 d) Decay rate is 20%
- 42a) 5 b) Growth c) Growth factor is 2 d) Growth rate is 100%
- 43a) 110 b) Growth c) Growth factor is 1.3 d) Growth rate is 30%
- 44a) 60 b) Decay c) Decay factor is 0.7 d) Decay rate is 30%
- 45) $y = 110(1.035)^x$ 46) $y = 1504(1.2)^x$ 47) $y = 63(0.72)^x$
- 48) $y = 545(0.996)^x$ 49) $y = 79(2.19)^x$
- 50a) Exponential equation b) Exponential equation c) Linear equation
- 51a) $N = 73t + 970$ b) $N = -36t + 970$ c) $N = 970(0.906)^t$
- d) $N = 970(1.061)^t$ e) $N = 970$
- 52a) $P = 1944(1.052)^t$ b) 2,772.08

53a) $P = 17000(0.92)^t$

b) 14,388.8

54) \$30,240.96

55a) $A = 1000(1.04)^t$

b) $A = 1000(1.04)^{t/2}$

c) $A = 1000(1.04)^{2t}$ (where t is number of years since initial investment)

d) $A = 1000(1.04)^{4t}$ (where t is number of years since initial investment)

56a) $A = 7400(1.036)^t$

b) $A = 7400(0.98)^t$

c) $A = 7400(1.11)^{t/3}$

d) $A = 7400(0.95)^{2t}$

57) $A = 150000(1.11)^{t/2}$

58) $A = 200000(1.015)^{t/5}$

59) $A = 195000(0.96)^{t/3}$

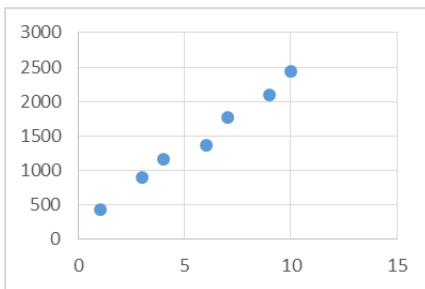
60) $A = 420000(0.92)^{t/4}$

Chapter 4

1a) below

b) Increasing

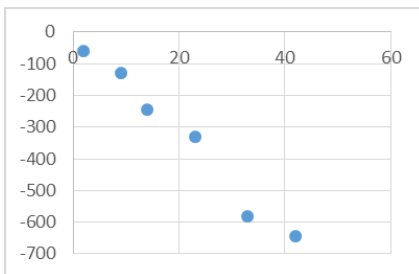
c) Yes



2a) below

b) Decreasing

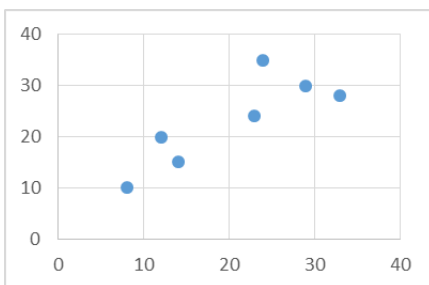
c) Yes



3a) below

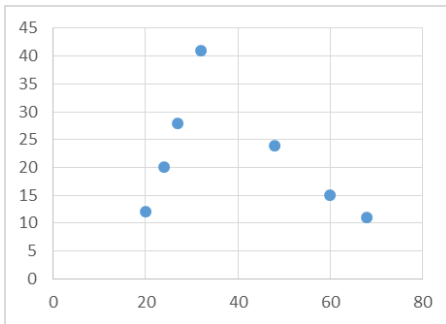
b) Increasing

c) Roughly linear



Appendix A

4a) below b) Both, neither one consistently c) No



5) $y = 213.3x + 232.4$

6) $y = -15.5x - 12.9$

7) $y = 0.79x + 7.10$

8) a) $r = 0.38$ b) $r = -0.85$ c) $r = -0.15$ d) $r = 0.94$

9) $r = 0.9933$. There is a strong, positive linear relationship between the two variables.

10) $r = -0.9890$. There is a strong, negative linear relationship between the two variables.

11) $r = 0.8323$. There is a moderately strong, positive linear relationship between the two variables.

12) $r = -0.3587$. There is a very weak, negative linear relationship between the two variables. No evidence of a linear relationship.

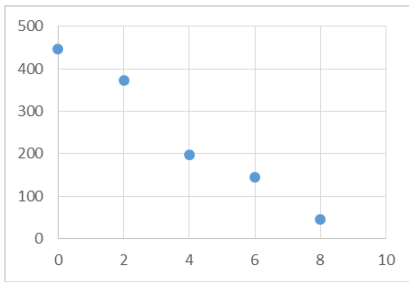
13) $r = -0.9872$. There is a very strong, negative linear relationship between the time and the total number of subscriptions. As the number of years since 2002 increases, the number of subscriptions decreases.

14) $r = 0.9965$. There is a very strong, positive linear relationship between the days spent hiking, and the distance he has hiked. As the number of days increases, the distance he has hiked also increases.

15) A correlation of -0.84 implies a stronger linear relationship than -0.47 . The closer the correlation is to -1 or 1 , the stronger the correlation. In this case -0.84 is closer to -1 than -0.47 is to -1 .

16) A correlation of -0.73 implies a stronger linear relationship than 0.51 . The closer the correlation is to -1 or 1 , the stronger the correlation. In this case, -0.73 is closer to -1 than 0.51 is to 1 .

17) a) below



b) Yes

c) $S = -51.65x + 448.2$

d) slope = -51.65 . For each additional year since 2002, the number of newspaper subscriptions decreases by approximately 51,650.

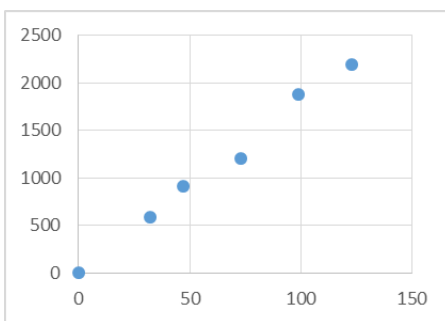
e) Vertical intercept = 448.2. This means that in 2002, the number of newspaper subscriptions was approximately 448,000.

f) $S = -51.65(5) + 448.2$, $S = 189.95$. In 2007, there will be approximately 189,950 subscriptions.

g) In 2004, there will be approximately 344,900 subscriptions. This value is close to, but not exactly equal to, the actual number from the data table. The regression equation does not fit each data point exactly, but instead provides an overall trend based on the entire data set.

h) No. The year 2030 is too far outside the original set of years (2002 – 2010). Using the regression equation to make predictions this far out is extrapolation and would result in poor predictions. In this case, you would be predicting a negative number of subscriptions!

18a) below



b) Yes

c) $D = 18.01t + 9.22$

d) slope = 18.01. Each day that Scott hikes, he goes approximately 18 additional miles.

e) $D = 18.01(50) + 9.22$, $D = 909.72$. After 50 days, Scott has hiked approximately 909.72 miles.

f) No. The hike is 2200 miles, so 2500 is not a possible output value from this equation.

Appendix A

19a) $P = 97.0(0.896)^t$ b) $r = 10.4\%$ c) 40.29

20a) $V = 3.73(1.071)^n$ b) 7.1%

c) 8.50, After 12 years, the value of the investment is approximately \$8,500

21a) $N = 2486.206(0.915)^t$ b) 8.5% decrease per day c) 1594.566

d) 1117.704, It is close to the value in the table, but not exact.

e) No. The regression equation is modeling the overall trend in the data and is used to make predictions and approximations.

22a) The data appear linear.



b) $S = 26.965a + 39.850$

c) $S = 1603.82$. If the company spends 58 million dollars in advertising, they can expect to have approximately 1603.82 million in sales.

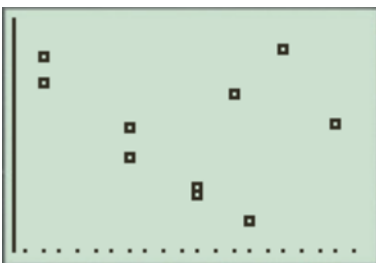
23a) The data appear exponential.



b) $S = 228.844(1.155)^m$

c) $S = 1289.803$. After 12 months, you should expect to have approximately 1290 subscribers to your YouTube channel.

24a) The data are neither linear nor exponential.



Chapter 5

1) \$210	2) \$77,040	3) \$23,700	4) \$42,120	5) \$2727.27	6) 27.8 years	7) 23.3 years
8) 19.6 years	9) \$488.67	10) \$3,869.68	11) \$11,458.55	12) \$2,498.90		
13) \$8,878.56	14) \$435.56	15a) \$3,641.51	15b) \$1,641.51	16a) \$27,137.65		
16b) \$17,137.65	17) \$7,140.96	18) \$9,250.64	19) \$6,624.12	20) \$2,964.38		
21) \$2,206.53	22) \$3,717.14	23) \$9,992.02	24) \$2,224.12	25) \$5,208.66		
26a) \$45,244.48	26b) \$39,100.00	27a) \$6,515.58	27b) \$6,000.00	28a) \$9,769.44		
28b) \$2.836%	29a) \$3,155.95	29b) 5.198%	30) 3.660%	31) 5.884%		
32) 4.783%	33) 2.320%	34) 5.9%	35a) 6.183%	35b) 6.168%		
35c) 6%	36a) 16.180%	36b) 16.075%	36c) 15%			

Chapter 6

1) \$41,645.91	2) \$75,387.07	3a) \$14,486.56	3b) \$10,000.00	3c) \$4,486.56
4a) \$29,081.87	4b) \$18,000.00	4c) \$11,081.87	5a) \$116,547.38	5b) \$72,000.00
5c) \$44,547.38	6a) \$658.44	6b) \$158,025.60	6c) \$341,974.40	7a) \$796.40
7b) \$286,704.00	7c) \$513,296.00	8) \$349.53	9) \$232,810.30	10) \$182,158.28
11) \$589,088.84	12a) \$320,243.29	12b) \$750,000.00	12c) \$429,756.71	
13a) \$401,447.24	13b) \$700,000.00	13c) \$298,552.76	14) \$3,582.16	
15) \$2,827.12	16) \$3,042.80	17a) \$130,397.13	17b) \$252,000.00	
17c) \$121,602.87	18a) \$12,625.50	18b) \$15,000.00	18c) \$2,374.50	
19a) \$194,346.78	19b) \$450,000.00	19c) \$255,653.22	20a) \$141,772.87	
20b) \$306,000.00	20c) \$164,227.13	21) \$546.72	22) \$863.16	
23a) \$180,000.00	23b) \$966.28	23c) \$1079.19	24a) \$1,618.79	
24b) \$282,764.40	25) \$542.38	26) \$974.29	27) \$354.15	
28a) \$265.61	28b) \$749.28	29) \$10,033.42	30) \$85,258.61	
31) \$76,634.11	32) \$67,283.33	33a) \$119,023.28	33b) \$118,973.37	
33c) \$108,103.28	33d) \$83,093.37			

Appendix A

33e) Lucy's account has more money overall and Lucy made more interest. Thus, Lucy made the best investment.

34a) \$129,414.07

34b) \$136,686.39

34c) \$116,454.07

34d) \$93,486.39

34e) Nick's account has more money overall. Robert made more interest. When considering who made more money off of a smaller amount of money, Robert made the better investment.

35) \$6,064.67

36) \$1,293.90

37) \$2,398.52

Chapter 7

1) {M, i, s, p}

2) {January, February, March, April, May, June, July, August, September, October, November, December}

3) The set of numbers 3, 6, and 9.

4) The set of letters a, e, i, o, u.

5) Yes

6) Yes

7) Yes

8a) true

8b) false

8c) true

8d) false

8e) false

8f) true

8g) false

9a) false

9b) true

9c) true

9d) true

9e) false

9f) true

9g) false

10) $B \subset A$

11a) true

11b) false

11c) true

11d) true

11e) false

11f) true

12) {1, 2, 3, 4, 7, 9, 10}

13) {orange, yellow, purple}

14) {2, 8, 12, 14}

15a) {2, 3, 5, 7, 10, 11, 12, 13, 14, 16, 17, 19, 20}

15b) {5, 10, 13, 14, 16, 19}

15c) {20}

16a) {1, 4, 5, 6, 8, 11, 12, 14, 15, 16, 17, 18}

16b) {11, 12, 16}

16c) {6, 14, 17, 18}

17a) {1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 19, 20}

17b) {9, 12, 14}

17c) {5, 17, 19,

18a) {2, 3, 5, 6, 8, 10}

18b) {} or $\emptyset$

18c) {5}

18d) {1, 4, 7, 9}

e) {1, 2, 4, 5, 6, 7, 9}

19a) {s, t}

19b) {c, k}

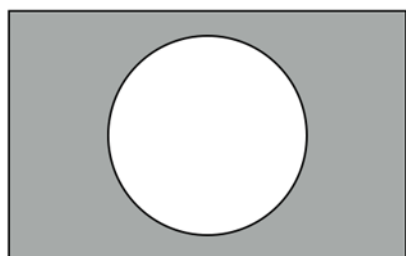
19c) {a, b, k, t, h}

19d) {a, b, k}

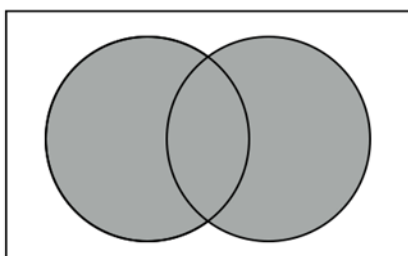
19e) {b, c, k}

19f) {h}

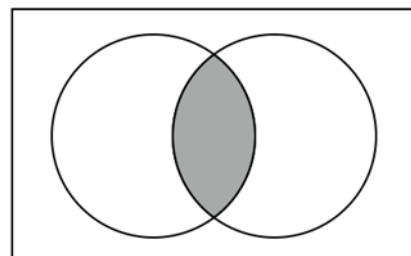
20)



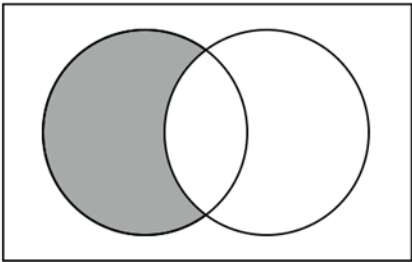
21)



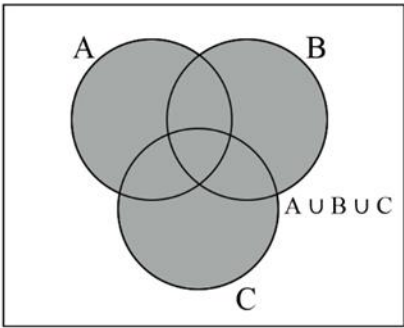
22)



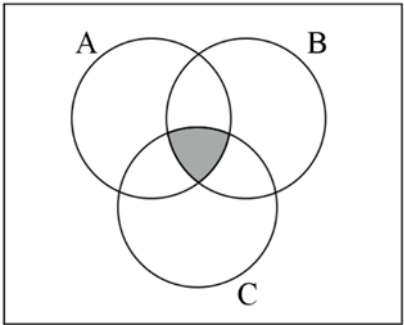
23)



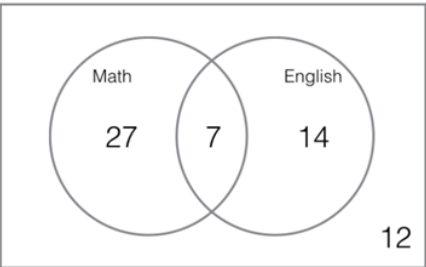
24)



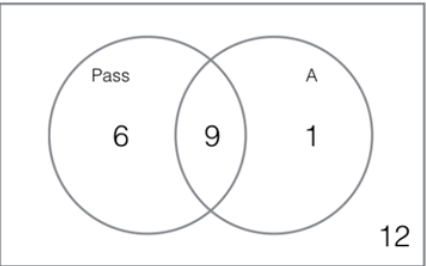
25)



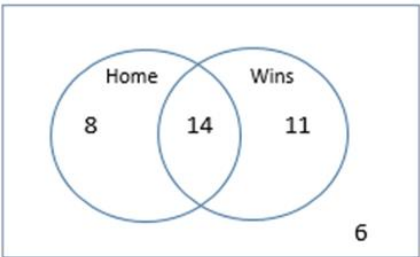
- 26a) 5 26b) 3 26c) 6 26d) 1 27a) 3 27b) 5 27c) 7
27d) 4 27e) 6 28a) 23 28b) 24 28c) 29 28d) 18 28e) 8
28f) 44 28g) 21 29a) 8 29b) 15 29c) 12 29d) 11 29e) 2
29f) 18 29g) 5 30a) 32 30b) 27 30c) 52 30d) 4 30e) 68
30f) 37 31) 43 32) 8 33) 22
34a) below 34b) 48 34c) 39 34d) 14



- 35a) below 35b) 13 35c) 6 35d) 12



- 36a) below 36b) 11 36c) 6



Appendix A

37a) 28

37b) 8

37c) 11

38a) 24

38b) 11

38c) 13

39a) 44

39b) 25

39c) 10

39d) 35

40) 136

41) 213

42a) 11%

42b) 44%

Chapter 8

1a) $\frac{1}{6}$

1b) $\frac{1}{2}$

1c) $\frac{2}{3}$

2) 0

3) $\frac{1}{3}$

4) $\frac{11}{12}$

5a) $\frac{2}{5}$

5b) $\frac{8}{13}$

6a) $\frac{1}{3}$

6b) $\frac{2}{3}$

7a) $\frac{6}{13}$

7b) $\frac{2}{13}$

8) $\frac{2}{37}$

9a) $\frac{21}{26}$

9b) $\frac{5}{26}$

10) $\frac{30}{67}$

11) $\frac{41}{81}$

12) $\frac{1}{13}$

13) $\frac{1}{4}$

14) $\frac{51}{52}$

15) $\frac{3}{4}$

16) $\frac{1}{13}$

17) $\frac{12}{13}$

18) $\frac{3}{13}$

19) $\frac{37}{61}$

20) $\frac{7}{45}$

21) $-\$0.19$

22) $\$0.03$

23) $-\$0.17$; No

24) $\$45.55$

25) $\$22.00$

26) $\$40.50$

Chapter 9

1) $\frac{1}{12}$

2) $\frac{1}{8}$

3) $\frac{1}{36}$

4) $\frac{1}{4}$

5) $\frac{1}{12}$

6) $\frac{1}{8}$

7) 4.41%

8) $\frac{8}{55}$

9a) $\frac{1}{169}$

9b) $\frac{12}{169}$

9c) $\frac{144}{169}$

9d) $\frac{1}{16}$

9e) $\frac{9}{13}$

10a) $\frac{1}{156}$

10b) $\frac{5}{48}$

10c) $\frac{3}{26}$

10d) 0

10e) $\frac{11}{13}$

11a) $\frac{2}{5}$

11b) $\frac{1}{3}$

12a) $\frac{1}{2}$

12b) $\frac{2}{3}$

13) $\frac{1}{2}$

14) $\frac{1}{6}$

15a) $\frac{3}{14}$

15b) $\frac{2}{5}$

16a) $\frac{5}{13}$

16b) $\frac{2}{11}$

16c) $\frac{1}{3}$

17a) $\frac{6}{17}$

17b) $\frac{2}{21}$

17c) $\frac{4}{17}$

18a) 0.0221

18b) 0.9997

19a) 0.0119

19b) 0.99998

20) $\frac{4}{663}$

21) $\frac{25}{102}$

22) $\frac{4}{663}$

23) $\frac{253}{9996}$

24) $\frac{1}{12}$

25) $\frac{17}{147}$

26) $\frac{2}{65}$

27a) $\frac{77}{300}$

27b) $\frac{77}{300}$

27c) $\frac{11}{60}$

27d) $\frac{91}{300}$

27e) $\frac{91}{300}$

28a) $\frac{1001}{6900}$

28b) $\frac{77}{690}$	28c) $\frac{1001}{6900}$	28d) $\frac{33}{460}$	28e) $\frac{91}{575}$	29a) $\frac{10}{65}$
29b) $\frac{8}{65}$	29c) $\frac{51}{65}$	29d) $\frac{44}{65}$	30a) $\frac{19}{81}$	30b) $\frac{10}{81}$
30c) $\frac{51}{81}$	30d) $\frac{67}{81}$	31) $\frac{5}{52}$	32) $\frac{2}{3}$	33) $\frac{4}{13}$
34) $\frac{11}{26}$	35) $\frac{7}{13}$	36) $\frac{1}{12}$	37) $\frac{5}{7}$	38) $\frac{6}{7}$
39a) $\frac{5}{8}$	39b) $\frac{3}{8}$	40) 0.67	41) 0.78	

Chapter 10

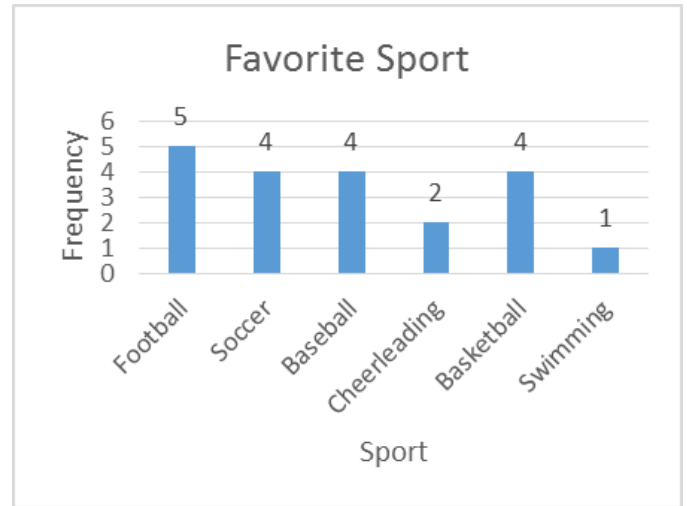
1) 60 outfits	2) 96 outfits	3) 450 outfits	4) 48 meals	
5) 480 one-topping pizzas	6) 67,600,000 license plates	7) 17,576,000 license plates		
8) 8,192	9) $26^8 = 208,827,064,576$ or 2.09×10^{11} passwords			
10) $62^{10} = 8.39 \times 10^{17}$ passwords	11) 100,000 code combinations	12) 4,194,304		
13a) 64	13b) 24	14a) 1,296	14b) 360	15a) 1,000
15b) 720	16) 24	17) 9,129,120	18) 840	19) 1.19×10^{13}
20) 495	21) 6,188	22) 15,890,700	23) 120	24) 2,598,960
25) 816	26) 208,606,320	27) 163,350		
28) $\frac{26 \times 25 \times 24 \times 23 \times 22 \times 21}{26^6} = 0.5366$	29) $\frac{5 \times 4 \times 3 \times 2}{26^4} = 0.0002626$			
30) $\frac{5^5}{26^5} = 0.0002630$	31) $\frac{1}{16 \times 15 \times 14 \times 13 \times 12} = \frac{1}{524160}$	32) $\frac{91}{17,383,860}$		
33) $\frac{252}{12,271,512}$	34) $\frac{12915}{12,271,515}$	35) $\frac{1287}{2,598,960}$	36) $\frac{48}{2,598,960}$	

Chapter 11

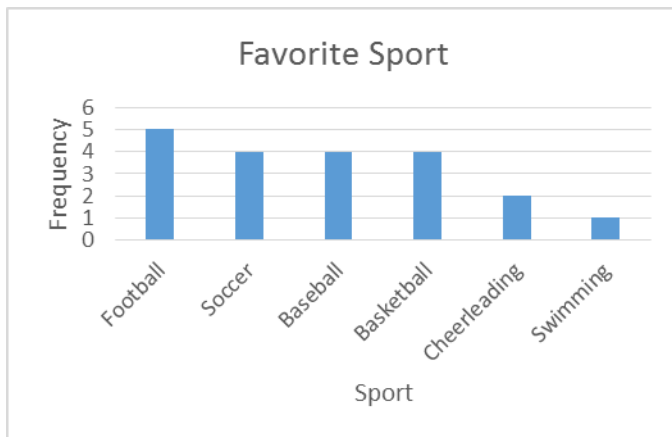
1a)

Sport	Frequency	Relative Frequency
Football	5	0.25
Soccer	4	0.2
Baseball	4	0.2
Cheerleading	2	0.1
Basketball	4	0.2
Swimming	1	0.05
Total	20	1

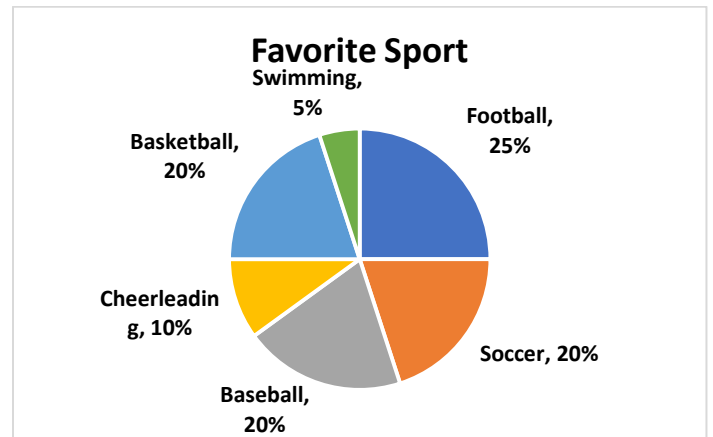
1b)



1c)



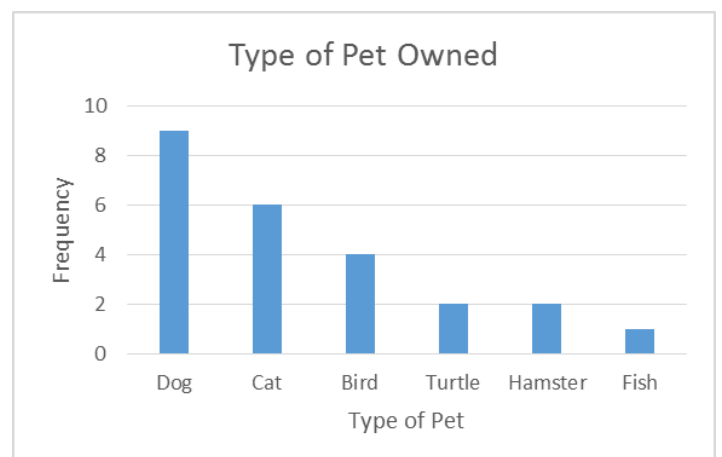
1d)



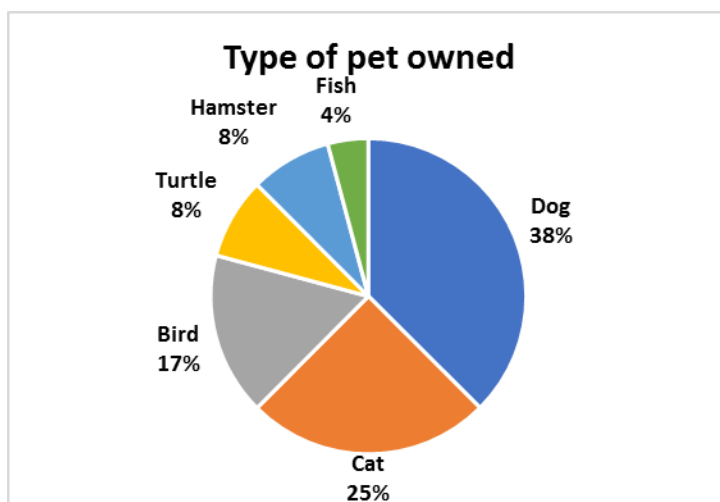
2a)

Type of Pet	Frequency	Relative Frequency
Dog	9	0.375
Cat	6	0.250
Bird	4	0.167
Turtle	2	0.083
Hamster	2	0.083
Fish	1	0.042
Total	24	1.000

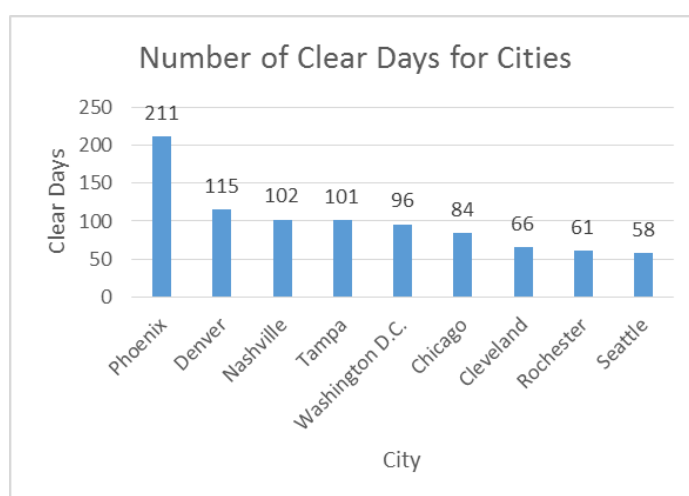
2b)



2c)



3)



4) $0.25(20) = 5$

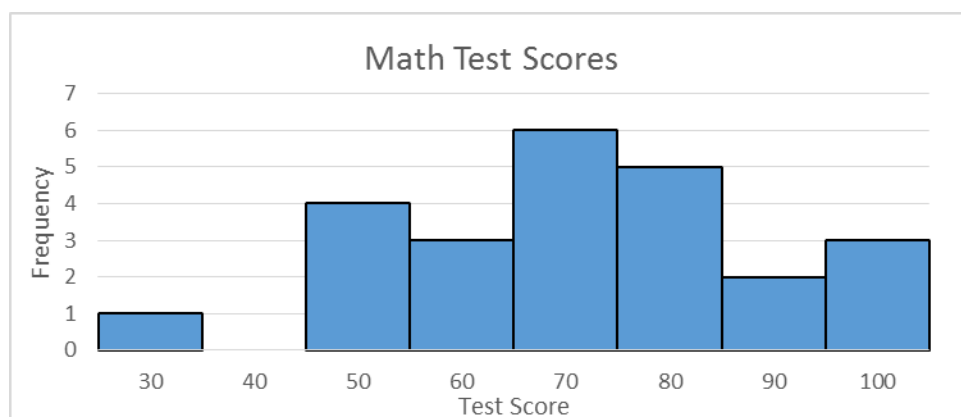
5) $0.26(\$2600) = \676

6) The graph needs a title, the axes need to be labeled (including units), and the 3D makes the graph difficult to read.

7a)

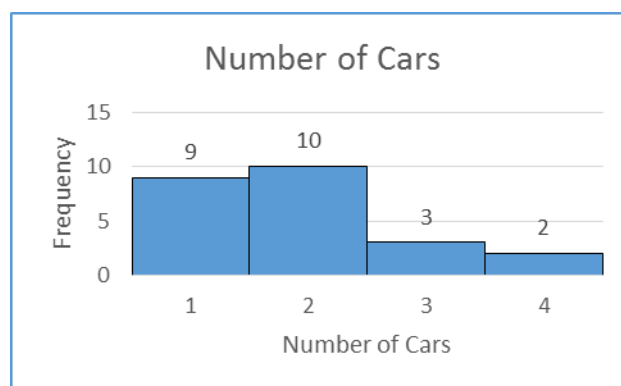
7b)

Test Score	Frequency
30	1
40	0
50	4
60	3
70	6
80	5
90	2
100	3



8a)

Number of Cars	Frequency	Relative Frequency
1	9	0.375
2	10	0.417
3	3	0.125
4	2	0.083



8b)

Appendix A

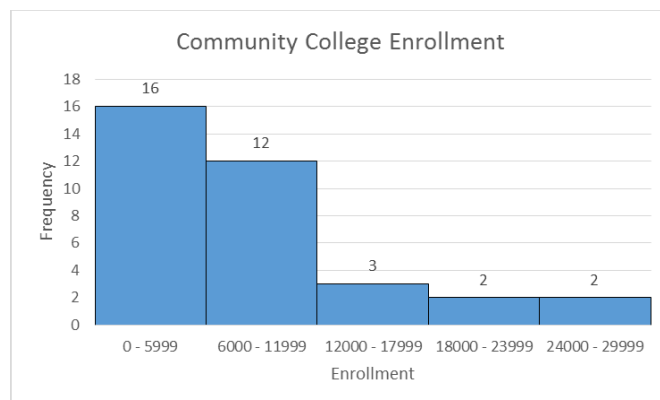
- 9a) 15 people 9b) 33.33% 10a) 19 orders 10b) about 21%
- 11a) Skewed 11b) slightly left-skewed 11c) No outliers
- 12a) Symmetric 12b) Bell shaped 12c) No outliers
- 13a) Skewed 13b) Left-skewed 13c) the exam score of 10 is an outlier.
- 14a) Symmetric 14b) Uniform 14c) No outliers
- 15a) Symmetric 15b) Bell shaped 15c) No outliers
- 16a) \$7.81 16b) \$7.75 16c) \$0.64
- 17a) 20.81 sec 17b) 19.95 sec 17c) 4.07 sec 18) 430 and 480
- 19a) the mode is "dog"
- 19b) the mode is the only measure of center useful for qualitative data.
- 20a) 0 20b) 1.47 20c) 1
- 21a) 2 21b) 2.26 21c) 2

22) Mean = \$129,400, Median = \$30,000. The median is a better measure of center for this data set. The mean is pulled toward the one extremely large annual salary. The median is a better representation of what the townspeople earn.

23a) \$338.40/credit hour 23b) \$316.04/credit hour

24a) Answers may vary 24b)

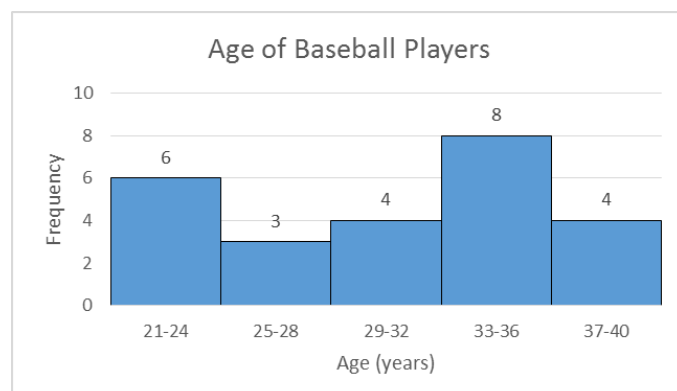
Enrollment	Frequency
0 - 5999	16
6000 - 11999	12
12000 - 17999	3
18000 - 23999	2
24000 - 29999	2



24c) 8628.7 24d) 6943.9

25a) 25b)

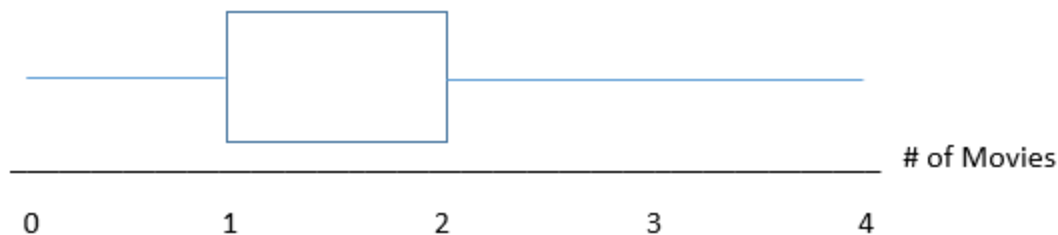
Age	Frequency
21-24	6
25-28	3
29-32	4
33-36	8
37-40	4



25c) 30.68 years 25d) 6.09 years

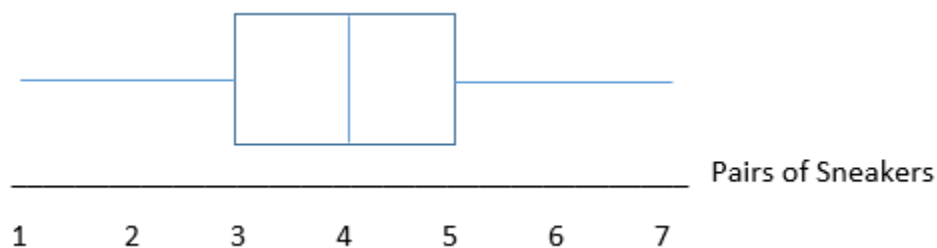
26a) 1.48 26b) 1.12 26c) 0, 1, 1, 2, 4

26d)



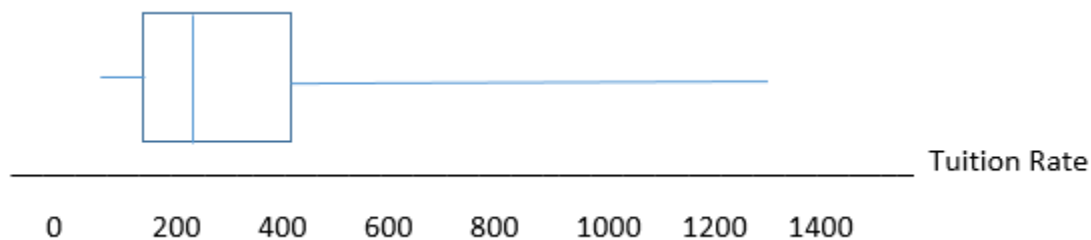
27a) 1, 3, 4, 5, 7

27b)



28a) \$87, \$175, \$218, \$416, \$1300

28b)



28c) \$175 is the 25th percentile

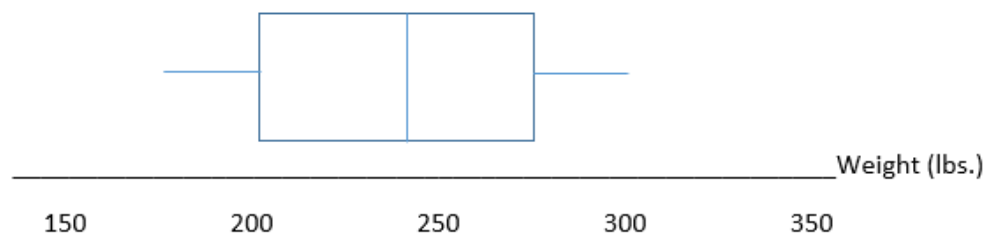
28d) \$218 is the 50th percentile

29a) 236.3 lbs.

29b) 37.9 lbs.

29c) 174, 206, 241, 272, 302

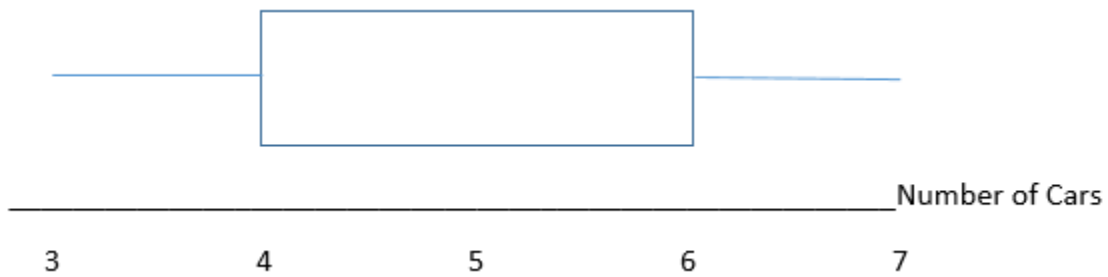
29d)



Appendix A

30a) 3, 4, 4, 6, 7

30b)



30c) the 50th percentile is 4

30d) the 75th percentile is 6

31a) 30, 40, 75, 90, 99 31b) 40, 75, 90, 94, 105

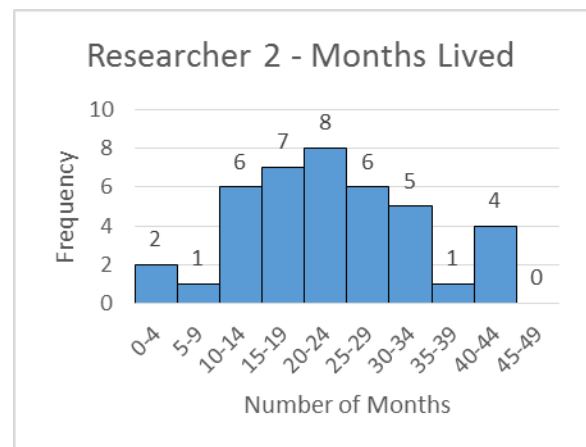
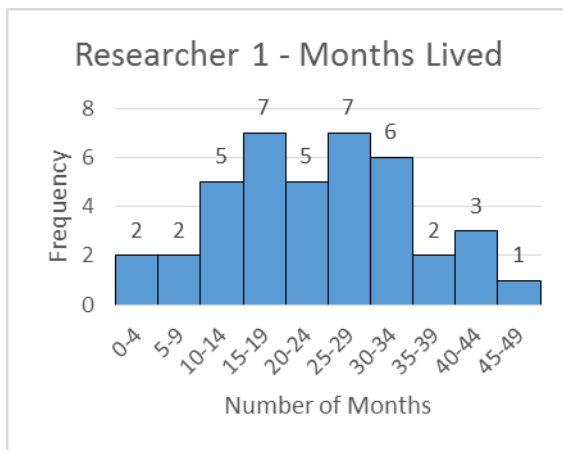
31c) Kendra makes more money, \$50 thousand more 32) 75%

33a) 7-series

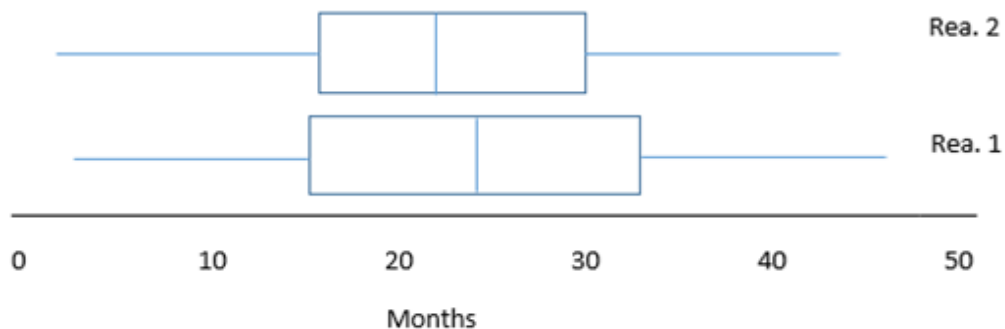
33b) 3-series

33c) No, the age increases with each series and people tend to make more money as they get older.

34a)



34b)

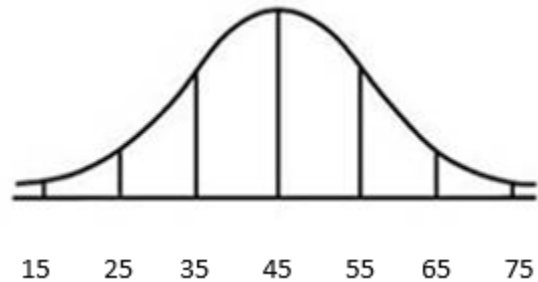
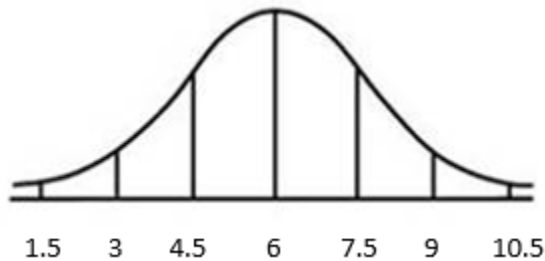


Chapter 12

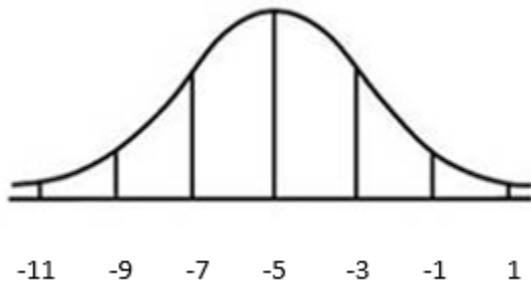
1) 61 2) 50%

3)

4)



5)



6) Distribution B has a wider spread along the horizontal axes, because it has a larger standard deviation. The spread of the distribution curve is determined by the standard deviation. The larger the standard deviation, the larger the spread of the distribution curve.

7) 68%

8) 95%

9) 4.7%

10) About 1 standard deviation above and below the mean. 68% of the data lies between 12 and 18.

11) About 2 standard deviations above and below the mean. 95% of the data lies between -5 and -1.

12) 49.85%

13) 34%

14) 27%

15) 31.7%

16) A z-score measures the number of standard deviations above or below the mean a particular observation lies.

17) 0; 1

18) $z = 2$ 19) $z = -1.5$ 20) $z = 2.78$ 21) $z = -0.133$ 22) $z = 1.5$ 23) $z = -1.67$ 24) $z = 0$ 25) $z = -0.33$

26) 1.25; left

27) 0.67; right

28) 6; right

29) 3.14; left

30) 1.7; left

31) $z = -0.85$

Appendix A

32a) $z = -0.51$. A basketball player who is 77 inches tall is 0.51 standard deviations below the mean height of 79 inches.

32b) $z = 1.54$. A basketball player who is 85 inches tall is 1.54 standard deviations above the mean height of 79 inches.

33a) Blood pressure of 100: $z = -1.79$, Blood pressure of 150: $z = 1.79$

33b) I would tell my friend that if his systolic blood pressure was 2.5 standard deviations below the mean, it would be 90. As we just calculated in part a) systolic blood pressure between 100 and 150 actually corresponds to between 1.79 standard deviations below the mean and 1.79 standard deviations above the mean.

34) $x = 20$

35) $x = 5.75$

36) $x = 7$

37) $x = 8$

38) 1.5 years

39a) An NBA player with a z-score of 3.5 would be 92.62 inches tall. That is just over 7 feet, 8 inches tall. While that height is possible, it's not very likely since it is more than 3 standard deviations above the mean. I probably would not believe him.

39b) An NBA player with a z-score of -3.5 would be 65.39 inches tall. That is just over 5 feet, 5 inches tall, which is less than 3 standard deviations below the mean. I would probably not believe him.

40a) A man with a z-score of 0.75 has a systolic blood pressure of 135.5. This is not an unusual blood pressure, as it's only 0.75 standard deviations more than the average blood pressure of 125.

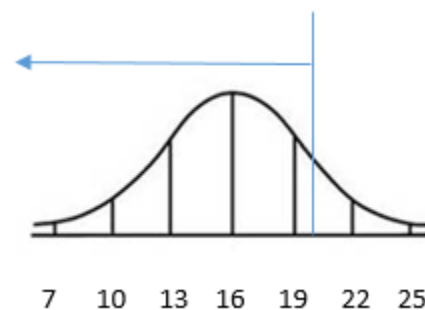
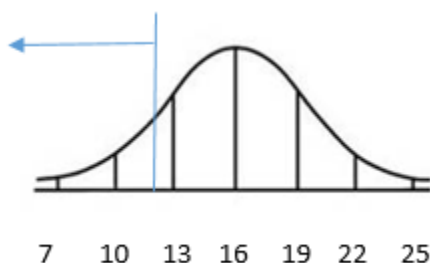
40b) A man with a z-score of 3.2 has a systolic blood pressure of 169.8. This is an unusual blood pressure, since it is more than 3 standard deviations above the mean.

41) $1 - 0.123 = 0.877$

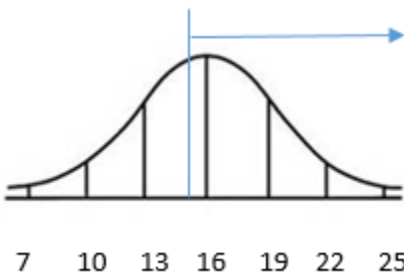
42) $1 - 0.543 = 0.457$

43a) 0.0912

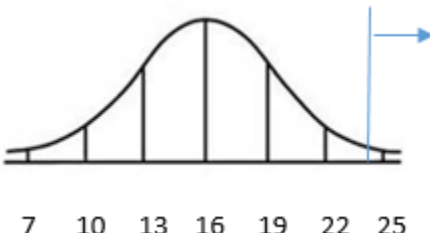
43b) 0.9088



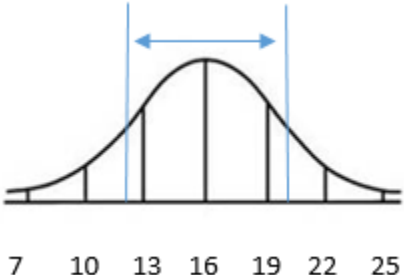
43c) 0.6306



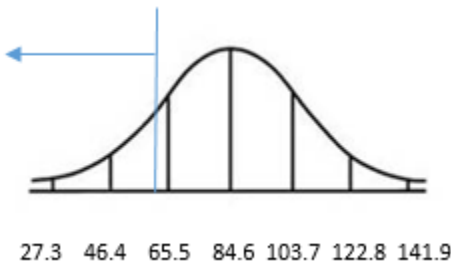
43d) 0.0038



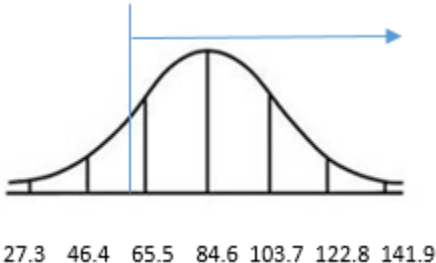
43e) 0.8176



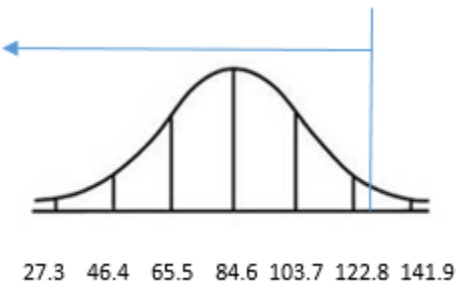
44a) 0.1035



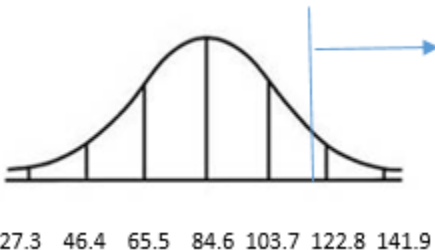
44b) 0.8965



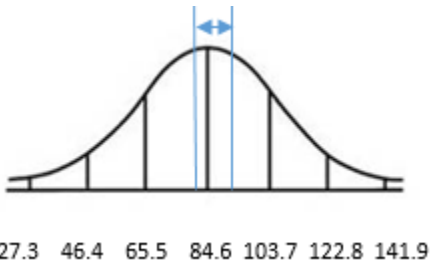
44c) 0.9885



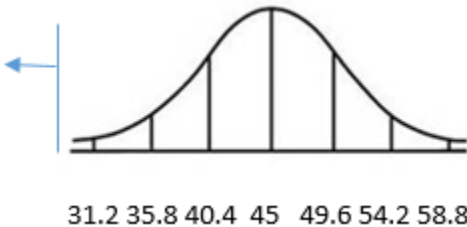
44d) 0.0346



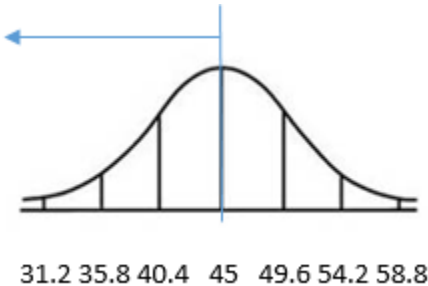
44e) 0.2065



45a) 0.000046

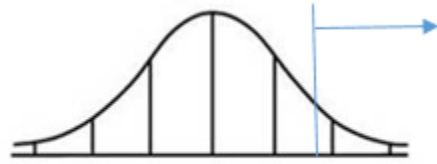


45b) 0.50



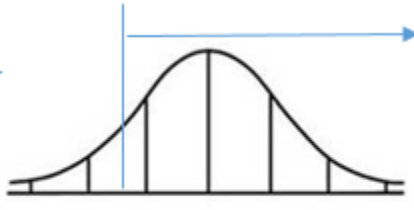
Appendix A

45c) 0.0640



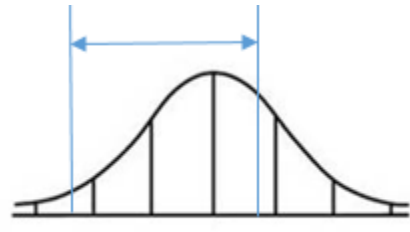
31.2 35.8 40.4 45 49.6 54.2 58.8

45d) 0.9360



31.2 35.8 40.4 45 49.6 54.2 58.8

45e) 0.7383



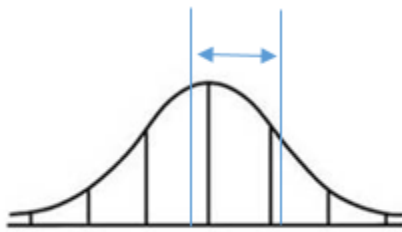
31.2 35.8 40.4 45 49.6 54.2 58.8

46a) 0.0580

46b) 0.9420

46c) 0.0126

47a) 0.5404



58.5 61 63.5 66 68.5 71 73.5

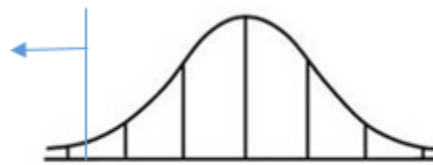
47b) A 9th grade boy who is 72 inches tall has a z-score of 2.4. The proportion of values greater than $z = 2.4$ is 0.0082. That means that it would be fairly unusual to see a 9th grade boy who is over 72 inches tall (only about 0.82% of 9th are over 72 inches tall).

48a) 0.0912



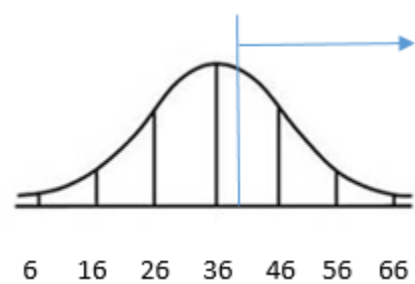
55 70 85 100 115 130 145

48b) 0.0038

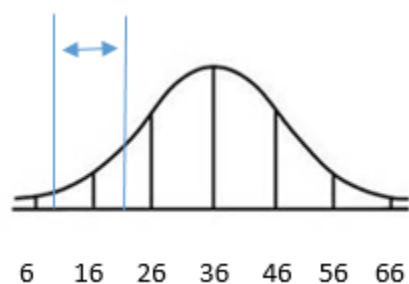


55 70 85 100 115 130 145

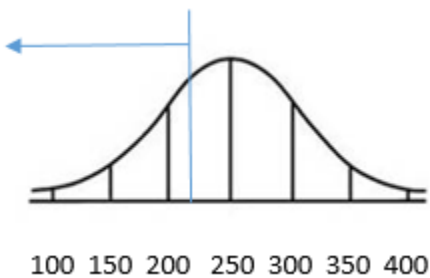
49a) 0.3446



49b) 0.0501



50a) 0.2743



50b) 0.0228

